Some Algorithms of Matrix Game Solutions

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INTRODUCTION

In practice we often deal with the problems which need to make decisions under conditions of uncertainty, i.e. there are situations when two (or more) sides have different purposes, and the results of any actions of each side depend on the activities of the partner. Such situations arise in a game of chess, checkers, dominoes, etc., refer to the conflict. The result of each player's stroke is dependent on a reciprocal course of the enemy, the goal - winning one of the partners. Conflict situations, arising in the economy are very frequent and have a diverse nature. These include, for example, the relationship between the supplier and the consumer, buyer and seller, the bank and the client. In all these examples, a conflict situation is generated by the difference between the interests of the partners and the desire of each of them to make optimal decisions that implement the goals the most. At the same time everyone has to reckon not only with their objectives but also with the objectives of the partner, and take into account the unknown in advance decisions that these partners will take.

For the literate solving conflict situations need scientifically sound methods. Such methods are developed the mathematical theory of conflict, which is called *game theory* [1].

In previous paper we were considered simple matrix games with two strategies of each player [2]. Here we try to explain how to solve matrix game of *higher dimension* without saddle points.

PRELIMINARIES

Game theory provides a mathematical framework for analyzing the decision-making processes and strategies of adversaries (or *players*) in different types of competitive situations. The simplest type of competitive situations are **two-person, zero-sum games.** These games involve only two players; they are called *zero-sum* games because one player wins whatever the other player loses [3].

Suppose that player A has m strategies $A_1, A_2, \ldots, A_m$ and player B has n strategies $B_1, B_2, \ldots, B_n$. This game has *mxn* dimension. As a result of selecting any pair of strategies by the players

A_i и B_j ($i=1, 2, \ldots m; j=1, 2, \ldots, n$) is uniquely determined the outcome of the game, i.e. Player A winning a_{ij} (positive or negative) and Player B losing ($-a_{ij}$). Assume that the values of a_{ij} are known for every pair of strategies (A_i, B_j). Then payoff matrix has theform.

B_i \ A_j	B_1	B_2	…	B_n
A_1	a_{11}	a_{12}	…	a_{1n}
A_2	a_{21}	a_{22}	…	a_{2n}
…	…	…	…	…
A_m	a_{m1}	a_{m2}	…	a_{mn}

Table 1. Payoff matrix for mxn game

The rows of the matrix correspond to the strategies of Player A, and columns - to the strategies of Player B.

Example 1

Director of the Transport company A, provides transportation services for passengers in the regional center, plans to open one or several routes: A_1, A_2,A_3 and A_4. For this purpose it was purchased 100 vans. It can supply all the traffic on one of the routes (most favorable) or spread over several routes. Demand for transport and, consequently, the company's profit depends on what routes in the near future will open by the main competitor - the company B. Its management fully controls the situation and can open some of the five routes B_1, B_2, B_3, B_4 и and B_5. Estimates of company A profit (th. Dollars) for any answer of company B provided by the payoffmatrix:

$$P = \begin{pmatrix} 5 & 6 & 4 & 6 & 9 \\ 6 & 5 & 3 & 4 & 8 \\ 7 & 6 & 6 & 7 & 5 \\ 6 & 7 & 5 & 4 & 3 \end{pmatrix}$$

Let's define "cautious" strategies for player A. Choosing optimal strategy A_i player A should rely on the fact that the player B response to this one of his strategies *Bj* for which the payoff for player A is minimal (player B seeks to " disserve" the player A).

Let's denote the smallest winning of player A as α_i when he choices strategy *Ai* for all possible strategies of the Player B. It is the smallest number in the *i*-th line of payoff matrix:

$\alpha_i = \min_j a_{ij}$.Then guaranteed winning of the player A for any strategy of the player B is equal to

$$\alpha = \max_i \min_j a_{ij}.$$

The value of $\alpha\alpha$ is called *lower value of the game*.

The player B is interested in reducing of the player A winning. Choosing a strategy Bj he considers the greatest possible gain for the A ($\beta_j = \max_i a_{ij}$). It the largest number in the j-th column of the payoff matrix.

Then a guaranteed losing Player B is equal to

$$\beta = \min_i \max_i a_{ij}.$$

The value of β is called upper value of the game.

A pair of strategies A_i and B_j gives an optimal solution if and only if the corresponding element aij is the largest in its column and the smallest in its row at the same time. Such a situation, if it exists, is called saddle point (similar to a saddle surface, which is curved upward in the same direction and curved down - in the other) [4].

If the game does not have a saddle point, the use of standard (pure) strategies does not give the optimal solution of the game. The search for such solutions makes it necessary to apply equalizing strategies: pure strategies alternate with some frequency [3].

Mixed strategies assume that each player will be randomly selected from valid possible pure strategies (but choose them with a certain probability), or partially implemented his pure strategies with defined proportions. Finding these probabilities (or ratios) is the solution of the game. Thus, in general terms, the decision of the game is a mixed strategy of players:

$$S_A = \begin{pmatrix} A_1 & A_2 & \dots & A_m \\ p_1 & p_2 & \dots & p_m \end{pmatrix}, S_B = \begin{pmatrix} B_1 & B_2 & \dots & B_n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$$

Here pi is probability of selecting pure strategy Ai by the player A; qj - probability of selecting pure strategy Bj by the player B (see example below). Obviously $\sum_i p_i = \sum_j q_j = 1$.

DOMINATION

The difficulty of the matrix games solution increases together with the dimension of the payoff matrix. Therefore games with large dimension of the payoff matrix can be simplified to reduce their dimension by eliminating duplicate and obviously unfavorable (dominated) strategies.

Definition 1. If all elements of the row (column) of game's payoff matrix equal to the corresponding elements of another row (column), the strategies, corresponding these lines (columns) called *redundant*.

Definition 2. If all the elements of the game's payoff matrix row, defining the strategy *Ai of the player A,* is not greater than the relevant elements of the other row, the strategy *Ai* is called *dominated* (obviously unfavorable).

Definition 3. If all the elements of the game's payoff matrix column, defining the strategy *Bi* of the player B, is not less than of the respective elements of another column, the strategy *Bi* is called *dominated* (obviously unfavorable).

Rule. The solution of the matrix game will not change if we deleted the rows and the columns of the payoff matrix, corresponding to the redundant and dominated strategies.

Example 2

Simplify payoff matrix of the game:

Ai \ Bj	B1	B2	B3	B4	B5
A1	5	9	3	4	5
A2	4	7	7	9	10
A3	4	6	3	3	9
→ A4	4	8	3	4	5
→ A5	4	7	7	9	10

Table 2. Payoff matrix 5x5 dimension

Solution

One can see from the payoff matrix that the strategy A5 doubles A2 strategy, so any of them can be discarded (we discard strategy A5). Comparing term by term strategy of A1 and A4, we see that each line item of A4 is not greater than the corresponding line item of A1. Therefore, the use by the player A dominating of over A4 strategy A1 always provides winnings, not less, than that which would be obtained when using the strategy A_4. Therefore, the strategy A_4 can be discarded. Thus, we have simplified game with the next payoff matrix:

B_j / A_i	B_1	B_2	B_3	B_4	B_5
A_1	5	9	3	4	5
A_2	4	7	7	9	10
A_3	4	6	3	3	9

Table 3. Reduced payoff matrix 3x5 dimension

From this matrix it can be seen that there are some strategies of the player B which are dominant over the other: B_2 is dominant over B_3, B_4 and B_5. Discarding dominated strategies B_2, B_4 and B_5, we get the game 3x2, which has the payoff matrix of the form:

B_j / A_i	B_1	B_3
A_1	5	3
A_2	4	7
A_3	4	3

Table 4. Reduced payoff matrix 3x2 dimension

In this matrix strategy A_3 is dominated by strategies A_1 and A_2. Discarding strategy A_3, we finally get a game with the payoff matrix 2x2 (see Tab.4). This game cannot be simplified, it should be solved discussed in [1] algebraicmethod.

B_j / A_i	B_1	B_3
A_1	5	3
A_2	4	7

Table 5. Finally reduced payoff matrix 2x2 dimension

Exercise 1

Find the optimal distribution of profits on the routes and the expected profits of transport companies in Example 1, making dimensionality reduction.

Solution

A_3>A_4 ⇒ *A_4* can be discarded. Reduced P takes a view

$$P_1 = \begin{pmatrix} 5 & 6 & 4 & 6 & 9 \\ 6 & 5 & 3 & 4 & 8 \\ 7 & 6 & 6 & 7 & 5 \end{pmatrix}.$$

For this matrix $B_2 \geq B_3$ and $B_1 \geq B_3 \Rightarrow B_1$ and B_2 can be discarded. Reduced P_1 takes a view

$$P_2 = \begin{pmatrix} 4 & 6 & 9 \\ 3 & 4 & 8 \\ 6 & 7 & 5 \end{pmatrix}.$$

For this matrix $A_1 \geq A_2 \Rightarrow$ A_2 can be discarded. Reduced P_2 takes a view

$$P_3 = \begin{pmatrix} 4 & 6 & 9 \\ 6 & 7 & 5 \end{pmatrix}.$$

For this matrix $B_4 \geq B_3 \Rightarrow B_4$ can be discarded. Finally reducing matrix has a view

$$P_4 = \begin{pmatrix} 4 & 9 \\ 6 & 5 \end{pmatrix}.$$

To solve the game with payoff matrix P4 let's check is it have saddle point first.

Recall formulas

$$p = \frac{c-d}{a-b+c-d},\ q = \frac{c-b}{a-b+c-d},\ v=\frac{ac-bd}{a-b+c-d}$$

for payoff matrix P= $\begin{pmatrix} a & b \\ d & c \end{pmatrix}$ [1].

So, $p_1=\frac{5-6}{4-9+5-6} = \frac{1}{6}$; $p_3=\frac{5}{6}$; $q_3=\frac{5-9}{4-9+5-6} = \frac{2}{3}$; $q_5= \frac{1}{3}$; $v=\frac{20-54}{4-9+5-6} = \frac{17}{3}$.

It means, that company A's optimal strategy is to allocate about 17% (1/6) to the route A_1 and about 83% (5/6) - to the route A_3. In turn, company B's optimal strategy is to allocate about 67% (2/3) to the route B_3 and about 33% (1/3) – to the route B_5. The expected profit of the company A is about \$5666.

Exercise 2

Generate Mathematica code, which allow to reduce the dimension of payoff matrix, if there is dominate strategies in the game. Use interactive input of data with help of InputFieldcommand.

Solution

See file Dimension reduct.nb

The solution of mxn games. Equivalent problems of Linear Programming

Theorem *1 If we add (subtract) to all elements of payment* *matrix the same number C, then the optimal mixed strategies of the players do not change, but the price of the game will increase (decrease) to the number C. [4]*

Let's we have a matrix game mxn dimension without a saddle point which has payoff matrix with elements a_{ij}. Assume that all winnings a_{ij} are positive (this can always be achieved due to the theorem 1 by adding to all the elements of the matrix sufficiently large number of C).

If all the a_{ij} are positive, so the value of the game for the optimal strategies is also positive, because $\alpha \le v \le \beta$.

According to the fundamental theorem of matrix games [3], if payoff matrix has no saddle point, there is a pair of optimal mixed strategies $S_A=(p_1, p_2, ..., p_m)$ and $S_B=(q_1, q_2, ..., q_n)$, the use of which provides players receive the win equals value of thegame.

Let us find S_A first. For this we assume that player B declined to its optimum mixed strategy SB and uses only pure strategies. In each of these cases, the winning player A will not be less than v:

$$\left.\begin{array}{l} a_{11}p_1 + a_{21}p_2 + \ldots + a_{m1}p_m \ge v, \\ a_{12}p_1 + a_{22}p_2 + \ldots + a_{m2}p_m \ge v, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}p_1 + a_{2n}p_2 + \ldots + a_{mn}p_m \ge v. \end{array}\right\} \qquad (1)$$

Dividing the left and right side of each of the inequalities (1) by the positive value of v and introducing the notations:

$$x_1 = \frac{p_1}{v}; \quad x_2 = \frac{p_2}{v}; \quad \ldots, x_m = \frac{p_m}{v}, \qquad (2)$$

we rewrite inequalities (1) in the next view

$$\left.\begin{array}{l} a_{11}x_1 + a_{21}x_2 + \ldots + a_{m1}x_m \ge 1 \\ a_{12}x_1 + a_{22}x_2 + \ldots + a_{m2}x_m \ge 1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{1n}x_1 + a_{2n}x_2 + \ldots + a_{mn}x_m \ge 1 \end{array}\right\}. \qquad (3)$$

Here $x_1, x_2, ... x_m$-non-negative variables.

By the fact that

$$p1+p2+...+pm=1,$$

variables $x_1, x_2, ... x_m$ satisfythecondition

$$x_1 + x_2 + \ldots + x_m = \frac{1}{v}. \qquad (4)$$

Taking into account that player A seeks to maximize his winning v, we obtain so called *linear programming problem* [5] : find a non-negative values of the variables $x_1, x_2, ... x_m$ such that they satisfy the linear constraints - inequalities (3) and minimize a linear function of these variables:

$$L(x)=x1+x2+...+xm\to min \qquad (5)$$

Using the solution of linear programming problem we find the value of the game *v* and the optimal strategy for SA by theformulas

$$v = \frac{1}{\sum_{i=1}^{m} x_i}, \qquad (6)$$

$$p_i = \frac{x_i}{\sum_{i=1}^{n} x_i} = x_i \cdot v, \; i = \overline{1, m}. \qquad (7)$$

Similarly we find the optimal strategy SB of the Player B. Suppose that player A renounced his optimal strategy SA and applies only pure strategies. Then losing of the player B in each of these cases, will be no greater than *v*:

$$\left.\begin{array}{l} a_{11}q_1 + a_{12}q_2 + \ldots + a_{1n}q_n \le v, \\ a_{21}q_1 + a_{22}q_2 + \ldots + a_{2n}q_n \le v, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}q_1 + a_{m2}q_2 + \ldots + a_{mn}q_n \le v. \end{array}\right\}. \qquad (8)$$

Dividing the left and right side of each of the inequalities (8) by the positive value of v and introducing the notations:

$$y_1 = \frac{q_1}{v}; \quad y_2 = \frac{q_2}{v}; \quad \ldots, y_n = \frac{q_n}{v}, \qquad (9)$$

we rewrite inequalities (8) in the view (10) with non-negative variables $y_1, y_2, ..., y_n$.

$$\left.\begin{array}{l} a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \le 1, \\ a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \le 1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}y_1 + a_{m2}y_2 + \ldots + a_{mn}y_n \le 1. \end{array}\right\} \qquad (10)$$

Basing on the fact, that $q1+q2+...+qn=1$, one can see that variables $y_1, y_2, ..., y_n$ satisfy condition

$$y_1 + y_2 + \ldots + y_n = \frac{1}{v}. \qquad (11)$$

Taking into account that player B tries to minimize positive price of v (his loss), we obtain the linear programming problem: to find a non-negative values of the variables $y_1, y_2, ..., y_n$ such that they satisfy the linear constraints - inequalities (10) and maximize linear function of thesevariables:

$$G(y)=y_1+y_2+ \ldots +y_n \to max. \qquad (12)$$

$$y_1, y_2, \dots, y_n \geq 0 \qquad (13)$$

The Player B's optimal strategy $S_B=(q_1, q_2, ..., q_n)$ is determined by using the solution of the linear programming problem (10), (12)-(13) by the formulas:

$$q_j = \frac{y_j}{\sum_{j=1}^{n} y_j} = y_j \cdot v, \quad j = \overline{1,n}. \qquad (14)$$

Thus, the optimal strategies SA=(p1, p2, ..., pm) and SB=(q1, q2, ..., qn) of mxn matrix game with the payoff matrix P={aij} can be found by solving a pair of dual linear programming

Problems [5]

Direct(default)problem

$$\min L(x) = \sum_{i=1}^{m} x_i ,$$

$$\sum_{i=1}^{m} a_{ij} x_i \geq 1, \; j = \overline{1,n};$$

$$x_i \geq 0, \; i = \overline{1,m}.$$

Dualproblem

$$\max G(y) = \sum_{j=1}^{n} y_j ,$$

$$\sum_{j=1}^{n} a_{ij} y_j \leq 1, \; i = \overline{1,m};$$

$$y_j \geq 0, \; i = \overline{1,n}.$$

Wherein

$$v = \frac{1}{\sum_{i=1}^{m} x_i} = \frac{1}{\sum_{j=1}^{n} y_j} = \frac{1}{\min L(x)} = \frac{1}{\max G(y)} \qquad p_i = x_i \cdot v; \quad i = \overline{1,m}; \quad q_j = y_j \cdot v; \quad j = \overline{1,n}$$

Example 3

Flatten the next matrix game to the linear programming problem

B_j / A_i	B_1	B_2	B_3
A_1	1	2	3
A_2	3	1	1
A_3	1	3	1

SOLUTION

1. Since α= 1 is not equal to β=3, then the game has no saddlepoint.

2. There is no duplication and dominated strategies in thisgame.

3. Mathematical models of a pair of dual problems of linear programming will be as follows:

Initial (direct) problem:

Find non-negative variables

$x_1, x_2, x_3,$

minimizing function

$L(x)=x_1 + x_2 + x_3 \rightarrow \min,$

with restrictions:

$$\begin{cases} x_1+3x_2+x_3 \geq 1; \\ 2x_1+x_2+3x_3 \geq 1; \\ 3x_1+x_2+x_3 \geq 1; \\ x_i \geq 0, \; i = \overline{1,3}. \end{cases}$$

Dual problem:

Find non-negative variables

$y_1, y_2, y_3,$

maximizing function

$G(y)=y_1 + y_2 + y_3 \rightarrow \max,$

with restrictions:

$$\begin{cases} y_1+2y_2+3y_3 \leq 1; \\ 3y_1+y_2+y_3 \leq 1; \\ y_1+3y_2+y_3 \leq 1; \\ y_j \geq 0, \; j = \overline{1,3}. \end{cases}$$

Example 4

Solve the game, given in the previous example with help of Mathematica using Linear Programming, Minimize, FindArgMax , FindMaxValue embedded functions.

Solution

See file Lin. prog.nb

Exercise 3

Two businessman A and B wish to conclude a tenancy agreement. Businessman A put 4 rental conditions A1, A2, A3, A4. At the same time, a businessman B may submit 5 variants of his claims for payment B1, B2, B3, B4, B5. Results of the expected profit for the company A (thous. Dollars) are presented in the matrix (see Table 5).

Find optimal strategies for both players and the profit estimates for the player A with help of Mathematica.

B_j / A_i	B_1	B_2	B_3	B_4	B_5
A_1	10	43	65	34	71
A_2	51	22	88	65	59
A_3	73	48	27	54	53
A_4	72	93	78	70	69

Table 6. Payoff matrix for the game “Rental of premises”

SOLUTION

1. Since $\alpha = 69$ is not equal to $\beta = 70$, so the game has no saddlepoint.
2. There is no duplication and dominated strategies in thisgame.
3. Mathematical models of a pair of dual problems of linear programming will be as follows:

Initial (direct) problem:

Find non-negative variables

x_1, x_2, x_3, x_4

minimizing function

$$L(x) = x_1 + x_2 + x_3 + x_4 \rightarrow \min,$$

with restrictions:

$$\begin{cases} 10x_1 + 51x_2 + 73x_3 + 72x_4 \geq 1; \\ 43x_1 + 22x_2 + 48x_3 + 93x_4 \geq 1; \\ 65x_1 + 88x_2 + 27x_3 + 78x_4 \geq 1; \\ 34x_1 + 65x_2 + 54x_3 + 70x_4 \geq 1; \\ 71x_1 + 59x_2 + 53x_3 + 69x_4 \geq 1; \\ x_i \geq 0, \ i = \overline{1,4}. \end{cases}$$

Dual problem:

Find non-negative variables

y_1, y_2, y_3, y_4, y_5

maximizing function

$$G(y) = y_1 + y_2 + y_3 + y_4 + y_5 \rightarrow \max,$$

with restrictions:

$$\begin{cases} 10y_1 + 43y_2 + 65y_3 + 34y_4 + 71y_5 \leq 1; \\ 51y_1 + 22y_2 + 88y_3 + 65y_4 + 59y_5 \leq 1; \\ 73y_1 + 48y_2 + 27y_3 + 54y_4 + 53y_5 \leq 1; \\ 72y_1 + 93y_2 + 78y_3 + 70y_4 + 69y_5 \leq 1; \\ y_j \geq 0, \ j = \overline{1,5}. \end{cases}$$

To solve these problems we use the code, generated for the previous exercise (see file Ex. 3.nb)

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