Computing Multiple Integrals Involving Matrix Exponentials

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ABSTRACT

In this paper, a generalization of a formula proposed by Van Loan [Computing integrals involving the matrix exponential, IEEE Trans. Automat. Control 23 (1978) 395–404] for the computation of multiple integrals of exponential matrices is introduced. In this way, the numerical evaluation of such integrals is reduced to the use of a conventional algorithm to compute matrix exponentials. The formula is applied for evaluating some kinds of integrals that frequently emerge in a number classical mathematical subjects in the framework of differential equations, numerical methods and control engineering applications.

Introduction

This note manages the calculation of various integrals including lattice exponentials, which often rise in various traditional scientific subjects in the structure of differential conditions, numerical strategies, control designing applications, and so

$$\int_0^1 \int_0^{S_1} \dots \int_0^{S_{k-2}} e^{A_{11}(t-S_1)} A_{12} e^{A_{22}(S_1-S_2)} A_{23} \dots e^{A_{kk}S_{k-1}} ds_{k-1} \dots ds_1 \qquad (1)$$

forth. In particular, basic of the frame

will be considered, where A_{ik} are $d_i \times d_k$ steady matrix, t>0 and k=1,2,...

Initially, the logical estimation of these integrals is by all accounts troublesome. In any case, in [19] a straightforward express recipe as far as certain exponential grid was given. In that way, various unmistakable integrals, for example

$$\int_0^t \mathrm{e}^{\mathbf{A}_{11}s} \mathbf{A}_{12} \mathrm{e}^{\mathbf{A}_{22}s}\, \mathrm{d}s$$

and

$$\int_0^t \int_0^{s_1} \mathrm{e}^{\mathbf{A}_{11}s_1} \mathbf{A}_{12} \mathrm{e}^{\mathbf{A}_{22}(s_1-s_2)}\, \mathrm{d}s_2\, \mathrm{d}s_1$$

can without much of a stretch be figured as specific instances of the said recipe [19]. In any case, such recipe is confined to integrals with assortment k≤4, which clearly restrains its convenience run.

The proposition of this note is summing up the recipe presented in [19] for any positive estimation of k. This has been emphatically roused for the need of figuring the integrals

$$\int_0^t e^{\boldsymbol{F}(t-s)} \boldsymbol{G}(s) ds \qquad (2)$$

and

$$\int_0^t e^{\boldsymbol{F}(t-s)} \boldsymbol{G}(s) \boldsymbol{G}^{\boldsymbol{T}}(s) e^{F^{\boldsymbol{T}}(t-s)} ds \qquad (3)$$

where F is a r×r steady framework and G: R→Rr×d a smooth capacity. Integrals of the sort (2) show up as a term in the explanatory arrangement of direct time-depending control system [16], straight common and postpone differential conditions [6]. Integrals like (3) are related to the thoughts of controllability and perceptibility Grammians of straight control systems [16]. They additionally show up as the covariance grid of the arrangement of straight stochastic differential conditions with time-depending added substance noise [1] and, with regards to separating hypothesis, as the framework commotion covariance lattice of the expanded Kalman channel for consistent discrete state-space models with added substance noise[10]. What›s more, they emerge in various numerical plans for the mix of conventional differential conditions that depend on polynomial approximations for the rest of the variety of consistent formula [9, 8, 7, 3, 4].

The paper has three segments. In the first, the summed up recipe is inferred, while in the second one the equation is connected to the calculation of the integrals (2) and (3). Last segment manages some computational angles for executing such an equation.

Main Result

A straightforward approach to figure single, twofold and triple integrals of the shape is provided [19, Theorem 1]:

$$\mathbf{B}_{12}(t) \equiv \int_0^t e^{\mathbf{A}_{11}(t-u)}\mathbf{A}_{12}e^{\mathbf{A}_{22}u}\,du,$$

$$\mathbf{B}_{13}(t) \equiv \int_0^t \int_0^u e^{\mathbf{A}_{11}(t-u)}\mathbf{A}_{12}e^{\mathbf{A}_{22}(u-r)}\mathbf{A}_{23}e^{\mathbf{A}_{33}r}\,dr\,du$$

and

$$\mathbf{B}_{14}(t) \equiv \int_0^t \int_0^u \int_0^r e^{\mathbf{A}_{11}(t-u)}\mathbf{A}_{12}e^{\mathbf{A}_{22}(u-r)}\mathbf{A}_{23}e^{\mathbf{A}_{33}(r-w)}\mathbf{A}_{34}e^{\mathbf{A}_{44}w}\,dw\,dr\,du,$$

as far as a solitary exponential lattice, yet not integrals of the shape (1) with $k \geq 5$. Next hypothesis defeats this confinement

Theorem 1. Let $d_1, d_2, \ldots, d_n$, be positive integers. If the n×n piece triangular lattice $A=[(Alj)]_{l,j=1:n}$ is characterized by

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \ldots & \mathbf{A}_{1n} \\ \mathbf{0} & \mathbf{A}_{22} & \ldots & \mathbf{A}_{2n} \\ \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{nn} \end{pmatrix},$$

Where (A_{lj}), l, j=1,...,n are $d_l \times d_j$ matrices such that $d_l = d_j$ for l=j, then for t≥0

$$e^{\mathbf{A}t} = \begin{pmatrix} \mathbf{B}_{11}(t) & \mathbf{B}_{12}(t) & \ldots & \mathbf{B}_{1n}(t) \\ \mathbf{0} & \mathbf{B}_{22}(t) & \ldots & \mathbf{B}_{2n}(t) \\ \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{nn}(t) \end{pmatrix},$$

With

$$\mathbf{B}_{ll}(t) = e^{\mathbf{A}_{ll}t}, \quad l = 1, \ldots, n, \tag{4}$$

$$\mathbf{B}_{lj}(t) = \int_0^t \mathbf{M}^{(l,j)}(t, s_1)\,ds_1 + \sum_{k=1}^{j-l-1} \int_0^t \int_0^{s_1} \ldots \int_0^{s_k} \sum_{l<i_1<\cdots<i_k<j} \mathbf{M}^{(l,i_1,\ldots,i_k,j)}(t, s_1, \ldots, s_{k+1})\,ds_{k+1}\ldots ds_1,$$
$$l = 1, \ldots, n-1, \quad j = l+1, \ldots, n, \tag{5}$$

where for any multi-index $(i_1, \ldots, i_k) \in N^k$ and vector(s1,...,sk) ∈ R^k the matrices $M^{(i1,\ldots,ik)}$ (s1,...,sk) are defined by

$$\mathbf{M}^{(i_1,\ldots,i_k)}(s_1, \ldots, s_k) = \left(\prod_{r=1}^{k-1} e^{\mathbf{A}_{i_r i_r}(s_r - s_{r+1})}\mathbf{A}_{i_r i_{r+1}}\right) e^{\mathbf{A}_{i_k i_k} s_k}$$

Proof. This verification should be founded on entire enlistment systems. The personalities (4–5) hold for n=1 - 4 by Theorem 1 in [19]. Assume that they additionally hold for n=m ≥4. At that point, let us demonstrate that (4–5) hold for any (m+1)×(m+1) block triangular matrix $A=[(Alj)]_{l,j=1:m+1}$.

Let us rewrite the matrix A as the 2×2 block triangular matrix $A = [(Alj)]_{i,j=1:2}$ defined by

$$\widetilde{\mathbf{A}} = \begin{pmatrix} [(\mathbf{A}_{lj})]_{l,j=1:m} & [(\mathbf{A}_{l,m+1})]_{l=1:m} \\ \mathbf{0} & \mathbf{A}_{m+1,m+1} \end{pmatrix}$$

and let B(t)=[(Blj(t))]l,j=1:m+1 be the (m+1)×(m+1) block triangular matrix given by

$$\mathbf{B}(t) = e^{\mathbf{A}t} = e^{\widetilde{\mathbf{A}}t}.$$

$$[(\mathbf{B}_{l,j}(t))]_{l,j=1:m} = e^{\widetilde{\mathbf{A}}_{11}t}, \tag{6}$$

$$\mathbf{B}_{m+1,m+1}(t) = e^{\mathbf{A}_{m+1,m+1}t}, \tag{7}$$

$$[(\mathbf{B}_{l,m+1}(t))]_{l=1:m} = \int_0^t e^{\widetilde{\mathbf{A}}_{11}(t-s_1)}\widetilde{\mathbf{A}}_{12}e^{\mathbf{A}_{m+1,m+1}s_1}\,ds_1. \tag{8}$$

Since *AA*ŒŒ = [(*AAll*Š)]§,ŠÂŒ:¡ then (6) and (7) imply that

$$\mathbf{B}_{ll}(t) = e^{\mathbf{A}_{ll}t}, \quad l = 1, \ldots, m+1, \tag{9}$$

$$\mathbf{B}_{lj}(t) = \int_0^t \mathbf{M}^{(l,j)}(t, s_1)\,ds_1 + \sum_{k=1}^{j-l-1} \int_0^t \int_0^{s_1} \ldots \int_0^{s_k} \sum_{l<i_1<\cdots<i_k<j} \mathbf{M}^{(l,i_1,\ldots,i_k,j)}(t, s_1, \ldots, s_{k+1})\,ds_{k+1}\ldots ds_1,$$
$$l = 1, \ldots, m-1, \quad j = l+1, \ldots, m. \tag{10}$$

On the other hand, since

$$e^{\widetilde{A_{11}}(t-s_1)} = \left[\left(B_{l,j}(t-s_1)\right)\right]_{l,j=1:m} \text{ and } \widetilde{A}_{12} = \left[\left(A_{l,m+1}\right)\right]_{l=1:m}$$

the identity (8) implies the following expression for each block

$\left(B_{l,m+1}(t)\right), l = 1, \ldots, m$:

$$\mathbf{B}_{l,m+1}(t) = \sum_{j=l}^{m} \int_0^t \mathbf{B}_{l,j}(t-s_1)\mathbf{A}_{j,m+1}e^{\mathbf{A}_{m+1,m+1}s_1}\,ds_1$$

Which by (9) and (10) gives

$$\begin{aligned}\mathbf{B}_{l,m+1}(t) = &\int_0^t \mathbf{M}^{(l,m+1)}(t, s_1)\,ds_1 + \sum_{j=l+1}^{m} \int_0^t \int_0^{t-s_1} \mathbf{M}^{(l,j,m+1)}(t, s_1+s_2, s_1)\,ds_2\,ds_1 \\ &+ \sum_{j=l+2}^{m} \sum_{k=1}^{j-l-1} \int_0^t \int_0^{t-s_1} \int_0^{s_2} \ldots \int_0^{s_{k+1}} \sum_{l<i_1<\cdots<i_k<j} \\ &\times \mathbf{M}^{(l,i_1,\ldots,i_k,j,m+1)}(t, s_1+s_2, \ldots, s_1+s_{k+2}, s_1)\,ds_{k+2}\ldots ds_1.\end{aligned}$$

(11)

For each k=0,1,... the change of variables u1=s1+s2,u2=s1+s3 ,..., uk+1 = s1+sk+2, uk+2=s1 in each of the multiple integrals that appear in (11) yields

$$\begin{aligned}\mathbf{B}_{l,m+1}(t) = &\int_0^t \mathbf{M}^{(l,m+1)}(t, u_1)\,du_1 + \sum_{k=1}^{m-l} \int_0^t \int_0^{u_1} \ldots \int_0^{u_k} \sum_{l<i_1<\cdots<i_k<j} \\ &\times \mathbf{M}^{(l,i_1,\ldots,i_k,j)}(t, u_1, \ldots, u_{k+1})\,du_{k+1}\ldots du_1,\end{aligned}$$

which when combined with (9) and (10) shows that the identities (4) and (5) hold forn=m+1, and the proof concludes.

Now we are able to evaluate the integrals $B_{12}(t)$, $B_{13}(t)$, $B_{14}(t)$ and the integrals

$$\mathbf{B}_{1k}(t)=\int_0^t\int_0^{s_1}\cdots\int_0^{s_{k-2}} e^{\mathbf{A}_{11}(t-s_1)}\mathbf{A}_{12}e^{\mathbf{A}_{22}(s_1-s_2)}\mathbf{A}_{23}\ldots e^{\mathbf{A}_{kk}s_{k-1}}\,ds_{k-1}\ldots ds_1$$

For k≥5 as well, all of them computed by just a single exponential matrix. This is, by applying the theorem above follows that the blocks $B_{12}(t)$, $B_{13}(t)$,...,$B_{1k}(t)$are easily obtained from e^{tA}, with

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_{22} & \mathbf{A}_{23} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{33} & \ddots & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{A}_{k-1,k}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{kk}\end{bmatrix}$$

3. Applications of generalize formula

This area is given to register integrals of the frame (2) and (3) by methods for the hypothesis expressed in the past segment.

Initially, assume that G(s) is a r×d polynomial matrix capacity of degree p. That is,

$$\mathbf{G}(s)=\sum_{i=0}^{p}\mathbf{G}_i s^i.$$

This can be equalize with

$$\mathbf{G}(s)=\mathbf{G}_0+\int_0^s\mathbf{G}_1\,du+\sum_{i=2}^{p}\int_0^s\int_0^{u_1}\cdots\int_0^{u_{i-1}}(i!\mathbf{G}_i)\,du_i\,du_{i-1}\ldots du_1$$

Then, by Theorem 1 it is easy to check that

$$\int_0^t e^{\mathbf{F}(t-s)}\mathbf{G}(s)\,ds=\mathbf{B}_{1,p+2}(t),$$

Where

$$\mathbf{B}(t)=[(\mathbf{B}_{lj}(t))]_{l,j=1:p+2}=e^{\mathbf{A}t},$$

and the block triangular matrix $A=[(A_{lj})]_{l,j=1:p+2}$ is given by

$$\mathbf{A}=\begin{pmatrix}\mathbf{F} & p!\mathbf{G}_p & (p-1)!\mathbf{G}_{p-1} & \cdots & \mathbf{G}_1 & \mathbf{G}_0\\ \mathbf{0} & \mathbf{0}_{d\times d} & \mathbf{I}_d & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}_{d\times d} & \ddots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \mathbf{I}_d & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}_{d\times d} & \mathbf{I}_d\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0}_{d\times d}\end{pmatrix}.$$

In a similar way, it is obtained that

$$\int_0^t e^{\mathbf{F}(t-s)}\mathbf{G}(s)\mathbf{G}^{\top}(s)e^{\mathbf{F}^{\top}(t-s)}\,ds=\mathbf{B}_{1,2p+2}(t)\mathbf{B}_{11}^{\top}(t)$$

Where $B(t)=e^{At}$ and the matrix A in this case is given by

$$\mathbf{A}=\begin{pmatrix}\mathbf{F} & \mathbf{H}_{2p} & \mathbf{H}_{2p-1} & \cdots & \mathbf{H}_1 & \mathbf{H}_0\\ \mathbf{0} & -\mathbf{F}^{\top} & \mathbf{I}_d & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\mathbf{F}^{\top} & \ddots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & \mathbf{I}_d & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -\mathbf{F}^{\top} & \mathbf{I}_d\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -\mathbf{F}^{\top}\end{pmatrix}$$

with

$$\mathbf{H}_i=\sum_{l+j=i}(l!j!)\mathbf{G}_l\mathbf{G}_j^{\top},\quad i=0,\ldots,2p$$

In case that G: R→Rr×d be a function with p continuous derivatives it can be approximated by means of its truncated Taylor expansion. That is,

$$\mathbf{G}(s)\approx\sum_{i=0}^{p}\frac{\mathbf{G}_i}{i!}s^i.$$

where the coefficient Gi is the ith derivative of G at s=0. In this case, the above procedure to compute the integrals (12) and (13) for polynomial G will give a convenient way to approximate integrals like (2) and (3). In [15], such approximation for (3) was considered early and an upper bound for it was also given. However, no closed formula in terms of a simple exponential matrix was given for this approximation. Such is the main improvement of this paper in comparison with [15]. At this point it is worth remark up that such approximations have been successfully applied to the computation of the predictions provided by the Local Linearization filters for non-linear continuous-discrete state-space models[12,13], as well as in[2,14]for computing other types of multiple integrals involving matrix exponentials. This evidences the practical usefulness of the result achieved in this paper.

Computational aspects

Clearly the primary computational assignment on the down to earth utilization of Theorem 1 is the utilization of a proper numerical calculation to register lattice exponentials. For example, those in light of reasonable Padé approximations, the Schur deterioration or Krylov subspace strategies (see[18,17] for fantastic surveys on viable techniques to register network exponential). The decision of one of them will primarily rely upon the size and structure of the matrix A in Theorem1. Much of the time, it is sufficient to utilize the calculations created in [5], which exploit the extraordinary structure of the matrix A. For a high dimensional matrix A, Krylov subspace methods are strongly recommended. Nowadays, a number of professional mathematical softwares such as MATLAB 7.0 provide efficient and precise codes for computing matrix exponentials. Therefore, the numerical evaluation of the integrals under consideration can be carried out in an effective, accurate and simple way. In fact, some numerical experiments have been performed for an application of the classical result of Van Loan (i.e. integrals of type (1) for k≤4). Specifically, in [11] were compared different numerical algorithms for the evaluation of some integrals of type (1) with k≤4. Those results could be easily extrapolated to our general setting since such a comparison was mainly focused on the dimension of the block triangular matrix A.

REFERENCES

1. L. Arnold, Stochastic Differential Equations: Theory and Applications, Wiley Interscience Publications, New York, 1974.
2. F. Carbonell, J.C. Jimenez, R. Biscay, Weak local linear discretizations for stochastic differential equations: convergence and numerical schemes, J. Comput. Appl. Math. 197 (2006) 578– 596.
3. S.M. Cox, P.C. Matthews, Exponential time differencing for stiff systems, J. Comput. Phys. 176 (2002) 430–455.
4. H. de la Cruz, R.J. Biscay, F. Carbonell, T. Ozaki, J.C. Jimenez, A higher order local linearization method for solving ordinary differential equations, Appl. Math. Comput. 185 (2007) 197– 212.
5. L. Dieci, A. Papini, Pade approximations for the exponential of a block triangular matrix, Linear Algebra Appl. 308 (2000) 103–202.
6. R.D. Driver, Ordinary and delay differential equations, Springer, Berlin, 1977.
7. R.A. Friesner, L.S. Tuckerman, B.C. Dornblaser, T.V. Russo,A method for exponential propagation of large systems of stiff nonlinear differential equations, J. Sci. Comput 4 (1989) 327–354.
8. A. Iserles, Quadrature methods for stiff ordinary differential systems, Math. Comp. 36 (1981) 171–182.
9. R.K. Jain, Some A-stable methods for stiff ordinary differential equations, Math. Comp. 26 (1972) 71–77.
10. A.H. Jazwinski, Stochastic Processes and Filtering Theory, Academic Press, San Diego, 1970.
11. J.C. Jimenez, A simple algebraic expression to evaluate the local linearization schemes for stochastic differential equations, Appl. Math. Lett. 15 (2002) 775–780.
12. J.C. Jimenez, T. Ozaki, Linear estimation of continuous-discrete linear state space models with multiplicative noise, Systems Control Lett. 47 (2002) 91–101.
13. J.C. Jimenez, T. Ozaki, Local linearization filters for nonlinear continuous-discrete state space models with multiplicative noise, Internat. J. Control 76 (2003) 1159–1170.
14. J.C. Jimenez, L.M. Pedroso, F. Carbonell, V. Hernandez, Local linearization method for numerical integration of delay differential equations, SIAM J. Numer. Anal. 44 (2006) 2584– 2609.
15. J.C. Jimenez, P. Valdes, L.M. Rodriguez, J. Riera, R.J. Biscay, Computing the noise covariance matrix of the local linearization scheme for the numerical solution of stochastic differential equations, Appl. Math. Lett. 11 (1998) 19–23.
16. P.S. Maybeck, Stochastic Models, Estimation, and Control, Academic Press, New York, 1982.
17. C.B. Moler, C.F. Van Loan, Nineteen dubious methods for computing the matrix exponential, twenty-five years later, SIAM Rev. 45 (1) (2003) 3–49.
18. R.B. Sidje, EXPOKIT: software package for computing matrix exponentials, AMC Trans. Math. Software 24 (1998) 130–156.
19. C.F. Van Loan, Computing integrals involving the matrix exponential, IEEE Trans. Automat. Control 23 (1978) 395–404.