Nash Equilibriums for Bimatrix Games

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ABSTRACT

Game theory has been a subject of interest for decades due to the fact that the analysis of different strategies provides the best possible strategy for a given situation. Its application is endless from war to business. More specifically, the Nash equilibrium looks at a competitive situation which is more applicable in our world. After learning about game theory, I was instantly mesmerized by the simple, yet complex rules and analysis of different strategies. Specifically for this paper, I used Mathematica which enabled me to recreate the payoff matrices and analyze them for the Nash equilibrium looking at different situations.

1. INTRODUCTION

Like the French philosopher, Jean-Paul Sartre once said, "Life is Choice between Birth and Death." In our lives, we make countless decisions and choices based on multiple variables. This, the study of decision making, is also known as game theory. Whether one is aware of the concept of game theory or not, game theory is used all the time. Game theory can be used in games like chess and poker but it is also used in business and more complicated matters.

Although game theory is used ubiquitously, the first tangible evidence of game theory dates back to 1713 when James Waldegrave wrote a strategic solution about a card game le Her. However, it wasn't until John von Neumann that game theory became big. He was able to prove his minimax theorem which established that in zero-sum games, there are certain moves that allows the player to minimize their losses and maximize their gains. In game theory, this is one of the most important concepts.

Prisoner's dilemma is one of the most famous example used in game theory. Basically, in prisoner's dilemma, two criminal gets arrested and each prisoner is given a choice to either stay quiet, or confess that his partner committed the crime. If both prisoners remain quiet, both will only have to serve 1 year in prison. However, if prisoner A betrays B (while B remains quiet), prisoner A will be free but prisoner B will have to serve 3 years (and vice versa). Finally, if both prisoner betray each other, both of them will have to serve 2 years in prison. Although this seems like a quite simple scenario, there is a lots of variables in this situation. The best situation for both prisoners would be to betray and hope that the other doesn't, but if both prisoners betrays, it leads to the worst case scenario.

Among the countless components of game theory, this paper will look specifically into Nash equilibrium. Nash equilibrium is a strategic solution in a non-cooperative game involving more than one player. In Nash Equilibrium, all the players knows each other's strategy and does nothing because they know that nobody can benefit from making a change. The Nash Equilibrium comes from John Forbes Nash. John Nash was a mathematician who earned a PH.D degree with a paper on non-cooperative game which contained the main properties of Nash Equilibrium.

2. Nash Equilibriums

Before moving on, there are some terms that should be discussed. When a game has N number of players, it is called non-cooperative game (given that N≥2). Both players have their own set of strategies z, of strategies Zi with the payoff function Hi(z), where zZ is a situation defined on the set. Like the name suggest, in a non-cooperative game, players are competing with each other to earn the highest possible point or gain. Non-cooperative game falls under a bigger category known as the constant sum game, if there exists a constant C, which is

$$\sum_{i \in N} H_i(z) = C \text{ for all situations } z \in Z.$$

The non-cooperative two player game in addition with the non-zero sum is called the bimatrix game.

Let 1st player has m strategies A1, ..., Am 2nd player – has n strategies B1, ..., Bn Winnings of the 1st and 2nd player is set by the payoff matrices A=[aij]mxn ; B=[bij]mxn.

Example 1

For example, picture two companies competing with each other to sell their product. Corresponding tensor P (double matrix) has a view

$$P = \begin{pmatrix} (0,4) & (4,0) & (5,3) \\ (4,0) & (0,4) & (5,3) \\ (3,5) & (3,5) & (6,6) \end{pmatrix}.$$

Here first elements of pairs in P refer to the player A and second one – to the player B.

So, player's A payoff matrix is

$$A = \begin{pmatrix} 0 & 4 & 5 \\ 4 & 0 & 5 \\ 3 & 3 & 6 \end{pmatrix} \text{ and player's } B \text{ payoff matrix}$$

$$\text{is } B = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 4 & 3 \\ 5 & 5 & 6 \end{pmatrix}.$$

They can have up to three different modifications on a market and their products are measured in million dollars. Company A's strategies are represented as the payment matrix A, and the Company B's strategies are represented as payment matrix B. If company A chose the second modification while company B chooses the third modification, company A will make $ 5 million while company B will make $ 3 million. An important point that should betaken into consideration is that this case is a zero sum game, players will play with their optimal strategies. However, for non-antagonistic games, players will choose an optimal strategy for the whole group so all the players can benefit from the action. Thus, the solution to a non-cooperative game is to find an equilibrium situation.

2.1 Nash Equilibriums in Bimatrix Games

Specifically, this section of the article would be looking into Nash equilibrium. Under ai0,j0≥ ai,j0 (i=1,...,m); bi0,j0≥ bi0,j (j=1,...,n), the Nash equilibrium strategies stand. All equilibrium strategies relates to the concept of the saddle point. A saddle point can be viewed as the lowest point in the x axis while being the highest point on the y axis. This can be used in game theory because players in equilibrium situation try to minimize their maximum loss. In game theory, we have to look for the maximal element in matrix A and B. Then all pairs of the strategies (i,j) would be known as aij and bij in. equilibrium situations Now let's take a look at some examples:

Example 2

Let's find equilibrium situations in the game of Example 1.

$$\text{For the matrix A we have } A^* = \begin{pmatrix} 0 & 4^* & 5 \\ 4^* & 0 & 5 \\ 3 & 3 & 6^* \end{pmatrix},$$

$$\text{for the matrix B} - B^* = \begin{pmatrix} 4^* & 0 & 3 \\ 0 & 4^* & 3 \\ 5 & 5 & 6^* \end{pmatrix}.$$

As we see, there are two asterisks for element with indexes i=j=3.
So, Nash equilibrium for players in this game corresponds to issue 3-d modification of production. Expected profit of the both firms equals to $ 6 million.

Example 3

Let's generate simple code in Mathematica to help us find Nash equilibrium for given payoff tensor P with interactive input of payoff matrixes A and B

Exercise 1

Find equilibrium situations in the game, characterized by the tensor P:

$$P = \begin{pmatrix} (3,2) & (4,3) & (5,1) & (6,2) \\ (5,5) & (2,1) & (8,4) & (3,6) \\ (8,7) & (3,0) & (9,6) & (2,8) \end{pmatrix}$$

a. without computer;
b. with help of Mathematica

Solution

a. Let’s separate payoff matrixes of the players:

$$A = \begin{pmatrix} 3 & 4 & 5 & 6 \\ 5 & 2 & 8 & 3 \\ 8 & 3 & 9 & 2 \end{pmatrix},$$

$$B = \begin{pmatrix} 2 & 3 & 1 & 2 \\ 5 & 1 & 4 & 6 \\ 7 & 0 & 6 & 8 \end{pmatrix}.$$

Then we mark maximal elements:

$$A^* = \begin{pmatrix} 3 & 4^* & 5 & 6^* \\ 5 & 2 & 8 & 3 \\ 8^* & 3 & 9^* & 2 \end{pmatrix}; B^* = \begin{pmatrix} 2 & 3^* & 1 & 2 \\ 5 & 1 & 4 & 6^* \\ 7 & 0 & 6 & 8^* \end{pmatrix}.$$

As we see, there is one equilibrium situation (A1,B2).Corresponding winnings are 4 conditional units for the player A and 3 conditional units for the player B.

b. Let’s use the code, generating in Example 3. The changes are minimal (see Fig.2)

```
InputField[MatrixForm[Array[A, {3, 3}]], FieldSize → Automatic]
(* Interactive input of 1-st payoff matrix elements *)

{{0, 4, 5}, {4, 0, 5}, {3, 3, 6}}

Z = Array[A, {3, 3}]; MatrixForm[Z] (* Z iz matrix of vectors *)
(0 4 5)
(4 0 5)
(3 3 6)

InputField[MatrixForm[Array[B, {3, 3}]], FieldSize → Automatic]
(* Interactive input of 1-st payoff matrix elements *)

{{4, 0, 3}, {0, 4, 3}, {5, 5, 6}}

V = Array[B, {3, 3}]; MatrixForm[V] (* Z iz matrix of vectors *)
(4 0 3)
(0 4 3)
(5 5 6)

ZT = Transpose[Z]; MatrixForm[ZT] (* Prepare for find maximums *)
(0 4 3)
(4 0 3)
(5 5 6)

k1 = 0 k2 = 100; For[i = 1, i ≤ 3, i++,
  (* i will be a number of line in the matrixe Z and number of a column in the matrix V *)
 For [j = 1, j ≤ 2, j++; If[Z[[i, j]] == Max[ZT[[i]]], k1 = i];
   (* j will be a number of column in the matrixe Z and number of a line in the matrix V *)
  If[V[[j, i]] == Max[V[[j]]], k2 = j]]; If[k1 = k2, Print["i=", i, " j=", j,
    " - Nash equilibrium. Winnings of players are ", "A: ", Z[[i, j]], " B: ", V[[j, i]]]];
 k1 = -10;
 k2 = -1]

i=3 j=3 - Nash equilibrium. Winnings of players are A: 6 B: 6
```

Figure 1. Mathematica code for find Nash equilibrium (Exercise 1 b).

2.2 Dominated strategies

By definition, dominated strategies are when the strategies I of Player A is greater than or equal to all elements of strategies J of Player B. In other words, a dominated strategy is when there is always an option of playing a better hand. However for this definition to stand, the solution to the bimatrix must be a constant even though we delete the rows and the columns of the matrix.

Example 4

a. Simplify payoff tensor P of the game and find its solution without.

$$P= \begin{matrix} & L & M & R \\ u & (4,3) & (5,1) & (6,2) \\ m & (2,1) & (8,4) & (3,6) \\ d & (3,0) & (9,6) & (2,8) \end{matrix}$$

```
InputField[MatrixForm[Array[A1, {3, 4}]], FieldSize → Automatic]
(* Interactive input of 1-st payoff matrix elements *)

3

Z = Array[A1, {3, 4}]; MatrixForm[Z] (* Z iz matrix of vectors *)
(0 4 5 6)
(4 0 5 3)
(3 3 6 2)

InputField[MatrixForm[Array[B1, {3, 3}]], FieldSize → Automatic]
(* Interactive input of 1-st payoff matrix elements *)

{{4, 0, 3}, {0, 4, 3}, {5, 5, 6}}

V = Array[B, {3, 3}]; MatrixForm[V] (* Z iz matrix of vectors *)
(4 0 3)
(0 4 3)
(5 5 6)

ZT = Transpose[Z]; MatrixForm[ZT] (* Prepare for find maximums *)
(0 4 3)
(4 0 3)
(5 5 6)

k1 = 0 k2 = 100; For[i = 1, i ≤ 3, i++,
  (* i will be a number of line in the matrixe Z and number of a column in the matrix V *)
 For [j = 1, j ≤ 2, j++; If[Z[[i, j]] == Max[ZT[[i]]], k1 = i];
   (* j will be a number of column in the matrixe Z and number of a line in the matrix V *)
  If[V[[j, i]] == Max[V[[j]]], k2 = j]];
 If[k1 = k2, Print["i=", i, " j=", j, " - Nash equilibrium. Winnings of players are ",
    "A: ", Z[[i, j]], " B: ", V[[j, i]]]];
 k1 = -10;
 k2 = -1]

i=3 j=3 - Nash equilibrium. Winnings of players are A: 6 B: 6
```

Figure 2. Mathematica code for find Nash equilibrium (Example 4).

Because player B's strategy R gives him/herself a definite win which is greater than the strategy M. Thus, strategy m is a dominated strategy and it is obvious that the rational player B would not play it. I.e. tensor P is reduced to P1:

$$P_1 = \begin{pmatrix} (4,3) & (6,2) \\ (2,1) & (3,6) \\ (3,0) & (2,8) \end{pmatrix}$$

However, if player A knows that player B will not use his strategy M, then his strategy U will be better than strategy M or D. I.e. tensor P1 is reduced to the line P2:

$$P_2 = ((4,3) \quad (6,2)).$$

On the other hand, if player B is aware that player A will play strategy Y, then he has to find an alternative strategy L. After the last reduction was the only element of the payment tensor - (4, 3). Thus optimal pair of strategies is (u,L). They give a win, equal to 4 conditional units to the player A and 3 conditional units to the player B.

b. Simplify payoff tensor P of the game and find its solution using interactive input of P with help of Mathematica.

Solution

```
{InputField[m, FieldSize → 5], InputField[n, FieldSize → 5]}
(* Interactive input of the payoff matrixes dimesion *)

{[3], [3]}

InputField[MatrixForm[Array[A, {m, n}]], FieldSize → Automatic]
(* Interactive input of the payoff matrix elements *)

[{{4, 5, 6}, {2, 8, 3}, {3, 9, 2}}]

a = Array[A, {m, n}]; MatrixForm[a]
(4 5 6
 2 8 3
 3 9 2)

InputField[MatrixForm[Array[B, {m, n}]], FieldSize → Automatic]
(* Interactive input of the payoff matrix elements *)

[{{3, 1, 2}, {1, 4, 6}, {0, 6, 8}}]

b = Array[B, {m, n}]; MatrixForm[b]
(3 1 2
 1 4 6
 0 6 8)

a = Array[A, {m, n}]; b = Array[B, {m, n}]; y = 0; x = Array[0 &, n]; z = x; v = y;
```

Figure 3. Mathematica code for payoff matrix reduction (Example 4). Part 1

Exercise 2

a. Simplify payoff tensor of the game using therule of dominated strategies deleting and find its solution without computer.

	gies	Player B					
		k	k	m	n	p	r
	a				)		)
	b			)			)
	c						
	d		)				

```
(* Organizing 1-st iteration loop for reducing rows of payoff matrixes A and B *)
For[i = 1, i < m, i++,
 For[k = i + 1, k < m + 1, k++,
  For [j = 1, j < n + 1, j++,
   If[TrueQ[a[[i, j]] ≥ a[[k, j]]], x[[j]] = 0, x[[j]] = 1]; y = y + x[[j]];
   If[TrueQ[b[[j, i]] ≥ b[[j, k]]], z[[j]] = 0, z[[j]] = 1]; v = v + z[[j]] ];
  If[y = 0, a = Drop[a, {k, k}];
   b = Drop[b, {k, k}];
   Print [k, "-th line of the matrixes A and B reducing"]];
  If[y = 3, a = Drop[a, {i, i};
     b = Drop[b, {i, i}]];
   Print [i, "-th line of the matrixes A and B reducing"]];
  If[v = 0, b = Drop[b, None, {k, k}]; a = Drop[a, None, {k, k}];
   Print [k, "-th column of the matrixes A and B reducing"]] ;
  If[v = 3, b = Drop[b, None, {i, i}];
   a = Drop[a, None, {i, i}];
   Print [i, "-th column of the matrixes A and B reducing"]]];
 y = 0;
 v = 0 ] ;
Print["Reducing payoff matrix A after excluding dominated rows ", MatrixForm[a]];
Print["Reducing payoff matrix B after excluding dominated rows ", MatrixForm[b]]

2-th column of the matrixes A and B reducing

Reducing payoff matrix A after excluding dominated rows (4 6
                                                         2 3
                                                         3 2)

Reducing payoff matrix B after excluding dominated rows (3 2
                                                         1 6
                                                         0 8)

{InputField[m1, FieldSize → 5], InputField[n1, FieldSize → 5]}
(* Interactive input of new payoff matrix dimesion *)

{[3], [2]}
```

Figure 4. Mathematica code for payoff matrix reduction (Example 4). Part 2

Solution to Exercise 2a)

Let's separate giving tensor on two matrixes, A and B:

$$A = \begin{pmatrix} 2 & -3 & -1 & -5 & 4 & -3 \\ 3 & 3 & -1 & 4 & 4 & -3 \\ 2 & 2 & 0 & 0 & -2 & -2 \\ -5 & -3 & 2 & -3 & 0 & 0 \end{pmatrix}; B = \begin{pmatrix} -3 & 2 & 2 & -3 & -5 & -5 \\ 1 & 3 & -1 & 4 & -4 & -4 \\ 1 & -3 & -5 & -5 & 3 & 0 \\ 4 & -5 & -1 & 3 & 1 & -2 \end{pmatrix}.$$

Beginning with the matrix B, one can see, that its last column B6 has the elements less or equal of the column's B5 corresponding elements. For brevity, we denote this fact as B5 ≥B6. So, reduced matrix B will have a view:

$$B1 = \begin{pmatrix} -3 & 2 & 2 & -3 & -5 \\ 1 & 3 & -1 & 4 & -4 \\ 1 & -3 & -5 & -5 & 3 \\ 4 & -5 & -1 & 3 & 1 \end{pmatrix}.$$

We should also reduce a matrix A by the same matter, so

$$A1 = \begin{pmatrix} 2 & -3 & -1 & -5 & 4 \\ 3 & 3 & -1 & 4 & 4 \\ 2 & 2 & 0 & 0 & -2 \\ -5 & -3 & 2 & -3 & 0 \end{pmatrix}.$$

There is a dominance situation: line A2 ≥ line A1. Thus, A1 and B1 can be reduced:

$$A2 = \begin{pmatrix} 3 & 3 & -1 & 4 & 4 \\ 2 & 2 & 0 & 0 & -2 \\ -5 & -3 & 2 & -3 & 0 \end{pmatrix}; B2 = \begin{pmatrix} 1 & 3 & -1 & 4 & -4 \\ 1 & -3 & -5 & -5 & 3 \\ 4 & -5 & -1 & 3 & 1 \end{pmatrix}.$$

One can see, that column B1≥ column B3, so next step of reduction leads to

$$A3 = \begin{pmatrix} 3 & 3 & 4 & 4 \\ 2 & 2 & 0 & -2 \\ -5 & -3 & -3 & 0 \end{pmatrix}; B3 = \begin{pmatrix} 1 & 3 & 4 & -4 \\ 1 & -3 & -5 & 3 \\ 4 & -5 & 3 & 1 \end{pmatrix}.$$

There is also dominance situation: line A1 ≥ line A2 and line A1 ≥ line A3. Therefore, we can reduce 2 lines at once:

$$A4 = (3 \quad 3 \quad 4 \quad 4); B4 = (1 \quad 3 \quad 4 \quad -4).$$

Finally, we choose maximum element in the matrix-line B4 and corresponding element of the matrix-line A4. Return to the initial tensor help us to determine optimal strategies of the players:

Ergo, solution of this game concerns a choice 2-nd strategy by the player A and 4-th strategy by the player B. Herewith they gain the same win, equal to 4 cond. u.

a. Simplify payoff tensor P of the game and find it's solution using interactive input of P with help of Mathematica.

a. Simplify payoff tensor of the game using therule of dominated strategies deleting and find its solution without computer.

Solution to Exercise 2 b)

To do this we use the code, generated for the Example 4 b).

This particular situation was when the payoff matrices were squares. However, of the situation contains a payoff matrix that is not a square, things become more difficult to compare. The cycles, generated for reducing of these matrices reduction, should be organized separately.

```
y = 0; x = Array[0 &, n]; z = x; v = y;
(* Organizing 2-nd iteration loop for reducing rows of payoff matrixes A and B *)
For[i = 1, i < m, i++,
 For[k = i + 1, k < m + 1, k++,
  For [j = 1, j < n + 1, j++,
   If[TrueQ[a[[i, j]] ≥ a[[k, j]]], x[[j]] = 0, x[[j]] = 1]; y = y + x[[j]];
    If[TrueQ[b[[j, i]] ≥ b[[j, k]]], z[[j]] = 0, z[[j]] = 1]; v = v + z[[j]] ;
          ]; If[y = 0, a = Drop[a, {k, k}];
   b = Drop[b, {k, k}];
   Print [k, "-th line of the matrixes A and B are reduced"]];
  If[y = 3, a = Drop[a, {i, i}];
     b = Drop[b, {i, i}]];
   Print [i, "-th line of the matrixes A and B are reduced"]];
  If[v = 0, b = Drop[b, None, {k, k}]; a = Drop[a, None, {k, k}];
   Print [k, "-th column of the matrixes A and B are reduced"]] ;
  If[v = 3, b = Drop[b, None, {i, i}];
   a = Drop[a, None, {i, i}];
   Print [i, "-th column of the matrixes A and B are reduced"]]];
 y = 0;
 v = 0] ;
Print["Reducing payoff matrix A after excluding dominated rows ", MatrixForm[a]];
Print["Reducing payoff matrix B after excluding dominated rows ", MatrixForm[b]]

Reducing payoff matrix A after excluding dominated rows (4 2 3)

Reducing payoff matrix B after excluding dominated rows (3 1 0)

{InputField[m2, FieldSize → 5], InputField[n2, FieldSize → 5]}
(* Interactive input of new payoff matrix dimesion *)

{3, 1}
```

Figure 5. Mathematica code for payoff matrix reduction (Example 4). Part 3

3.Mixed Strategies in 2x2 Bimatrix Games.

One of the most well-known example in game theory is called battle of the sexes (BoS) which is a two player coordination game. The backstory behind BoS is that a male and a female is planning to go on a date and they are deciding how to spend their time together. The male wants to go watch a soccer game while the female wants to go shopping. (The specific activities doesn't matter) But more importantly, both the male and the female want to go together. The following table shows the possible outcome.

		woman	
man		soccer	theatre
	soccer	(2 times happy, happy)	(not happy, not happy)
	theatre	(not happy, not happy)	(happy, 2 times happy)

This table shows that there are four possible outcome. The best options in this situation would be that the male and the female decide on the same thing while the worst options in this situation would be that they decide on different things. For example, if the man and the woman decide to watch the soccer game, the man is extremely happy because he gets to watch soccer and be with the women while the woman is happy because she gets to be together with the man. This game can be divided up into two cases.

1. With probability
2. Without probability

If the man and the woman make the same choice, and equilibrium is achieved, given the situation that they don't regret their own decisions. However, things get more complicated if we assume that the man and the women will choose their strategy/ activity based on some probability.

Definition 2 Mixed strategies of the players A and B in bimatrix game 2x2 are a set of probabilities X=(p,1-p), Y= (q,1-q)T, with which the players choose their pure strategies.

Here we suppose that the players have next payoff matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Definition 3 The result of multiplying matrix

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ by the column $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

is a column $D = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$, where $d_i = a_{i1}c_1 + a_{i2}c_2$ [9].

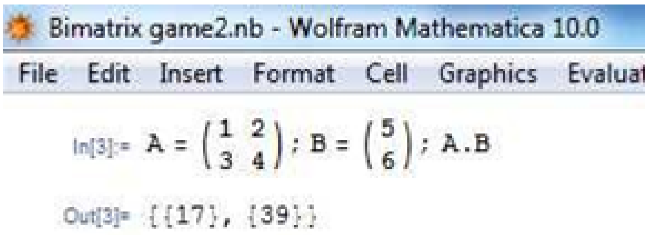
Example 5

Let's multiply $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ by $C = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$.

Due to rule, we have

$$AC = D = \begin{pmatrix} 1 \cdot 5 + 2 \cdot 6 \\ 3 \cdot 5 + 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 17 \\ 39 \end{pmatrix}.$$

This can performed in MATHEMATICA:



Exercise 3

Multiply $A = \begin{pmatrix} -1 & 2 \\ -3 & 0 \end{pmatrix}$ by $C = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$

a. "by hand"

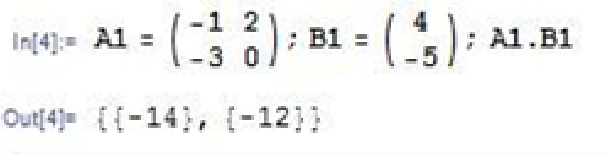
b. with help of MATHEMATICA

Solution

a. Due to rule, we have

$$AC = D = \begin{pmatrix} -1 \cdot 4 + 2 \cdot (-5) \\ -3 \cdot 4 + 0 \cdot (-5) \end{pmatrix} = \begin{pmatrix} -14 \\ -12 \end{pmatrix}$$

b.



Definition 4

The expected payoff of the players A and B are the values H1=X·(AY) and H2=(XB)·Y respectively.

Example 6

Let's define expected payoff in the game "family dispute", if both players choose their pure strategies with equal probabilities: X=(1/3,2/3); Y=(2/3,1/3).

Solution

First we write payoff matrices in numeric format:

$$A = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix},\ B = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}.$$

Then we calculate

$$H_1 = \left(\tfrac{1}{3}\ \tfrac{2}{3}\right)\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}2/3\\ 1/3\end{pmatrix}=\left(\tfrac{1}{3}\ \tfrac{2}{3}\right)\begin{pmatrix}4/3\\ 1/3\end{pmatrix}=4/9+2/9=2/3$$ [1].

And, by the similar way we get $H_2 =$

$$\left(\tfrac{2}{3}\ \tfrac{1}{3}\right)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1/3\\ 2/3\end{pmatrix}=\left(\tfrac{2}{3}\ \tfrac{1}{3}\right)\begin{pmatrix}1/3\\ 4/3\end{pmatrix}=2/9+4/9=2/3.$$

In MATHEMATICA one can do it as

```
X = (1/3 2/3); Y = (2/3
                    1/3); A2 = (2 0
                                0 1); B2 = (1 0
                                            0 2);
In[6]:= H1 = X.A2.Y
Out[6]= {{2/3}}
In[7]:= H2 = X.B2.Y
Out[7]= {{2/3}}
```

Definition 5: In essence, the Nash mixed equilibrium in the bimatrix game is a combination of mixed strategies of x and y where x is the most appropriate response to the strategy y and vice versa.

In other words, in a mixed equilibrium an action of an individual player changing his strategies alone does not bring profit to anyone. To demonstrate this definition we prove, that X=(1/3,2/3); Y=(2/3,1/3) is mixed equilibrium in the game "family dispute" or also known as Battle of Sexes as we already talked about in this paper.

Example 7

Let's denote pure strategies of the players as S (soccer game) and T (theatre) (or any stereotypical male and female activities); equilibrium mixed strategies P*=(p*,1-p*) for husband and Q*=(q*,1-q*) for wife.

Due to the definition 5 we have

$$H1(S, q^*) = H1(T, q^*), \qquad (2)$$

i.e. the win of the player 1 should not change, if he will play his pure strategy "S" or "T" instead of his mixed equilibrium strategy p* (provided player 2 – wife – plays her mixed equilibrium strategy q*).

Similar situation for the win of the second player:

$$H2(p^*,S) = H2(p^*,T), \qquad (3)$$

i.e. the win of the player 2 should not change, if she will play his pure strategy "S" or "T" instead of his mixed equilibrium strategy q* (provided player 1 – husband – plays his mixed equilibrium strategy p*).

Calculating H1(S, q*) and H1(T, q*) using formula (1) with S=(1,0) and T=(0,1) we get

$$H_1(S, q^*)=(1,0)\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}q^*\\ 1-q^*\end{pmatrix}=2\cdot q^*+0\cdot(1-q^*)=2q^*;$$

$$H_2(T, q^*)=(0,1)\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}q^*\\ 1-q^*\end{pmatrix}=0\cdot 2q^*+1\cdot(1-q^*)=1-q^*.$$

Equality (2) gives

$$2q^*=1-q^* \Rightarrow q^*=1/3.$$

Thus, Q*=(1/3,2/3). Analogically,

$$H_2(p^*,S)=(p^*,1-p^*)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\end{pmatrix}=p^*;$$

$$H_2(p^*,T)=(p^*,1-p^*)\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\cdot\begin{pmatrix}0\\ 1\end{pmatrix}=2-2p^*.$$

Equality (3) gives

$p^*=2-2p^* \Rightarrow p^*=2/3$. Thus, P*=(2/3,1/3).

Such equilibrium means, that each player should choose what he likes in two thirds of cases, and what likes his opponent – in one third of cases.

Exercise 4

Another example we can investigate is known as the "The struggle for markets".

In "the struggle for markets" there are essentially two players, player 1 (a small company) and player 2 (a bigger company).In this specific situation, player 1 wants to sell a large quantity of goods in one of the two markets controlled by another.To accomplish this, he has two options. The first option is that he can take one of the market (for example, to develop an advertising campaign).

In response to this certain strategy, the dominant player 2 might take precautionary measure to prevent this from happening. If player 1 doesn't encounter any obstacles, player 1 captures the market. However, if he encounters any obstacle, he is defeated. Selection markets by firms are their pure strategies.

Let the first market be more favorable for the player 1, but fighting for the first market requires a lot of budget. It is known that winning the first market would bring player 1 the double the profit compared from the second market. In the same sense, if player 1 loses the first market (his loss is 10) and player 2 gets rid of his competitor (his payoff is 2)

Described bimatrix game can be defined by the payoff matrices:

$$A = \begin{pmatrix} -10 & 2 \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}$$

where 1 unit is equal to $100 000. The task is

a. Find mixed equilibrium with help of formulas (2)-(3) with help of MATHEMATICA.

b. Calculate the expected payoff of the firms, if they choose their mixed equilibrium strategies "by hand".

Solution

a. Let's denote pure strategies of the players as I and II; equilibrium mixed strategies P*=(p*,1-p*) for the 1-st firm and Q*=(q*,1-q*) for the 2-nd firm.

Conditions of mixed equilibrium will have a form:
H1(I, q*) = H1(II, q*); H2(p*,I) = H2(p*,II).

Do the same procedure as in example 7, we get

$$H_1(I, q^*)=(1\ \ 0)\begin{pmatrix} -10 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} q^* \\ 1-q^* \end{pmatrix}=2-12q^*.$$

$$H_1(II, q^*)=(0\ \ 1)\begin{pmatrix} -10 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} q^* \\ 1-q^* \end{pmatrix}=-1+2q^*$$

So, H1(I, q*) = H1(II, q*) gives equation
2-12q*= 1+2q* ⇒ q*=3/14. Hence, Q*=(3/14;11/14).

Analogically,

$$H_2(p^*,I)= (p^*,1-p^*)\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}=-1+3p^*;$$

$$H_2(p^*,II)= (p^*,1-p^*)\begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}=1-3p^*.$$

So, H2(p*,I)= H2(p*,II) gives equation -1 +3p*=1-3p*⇒p*=1/3. Hence, P*=(1/3;2/3). This result have got by the next action in MATHEMATICA:

```
In[10]:= X3 = {0, 1}; A3 = ( -10  2 ); Y3 = (  q  ); X3.A3.Y3
                            (  1  -1 )       ( 1 - q )
Out[10]= {-1 + 2 q}

In[11]:= X4 = {1, 0}; X4.A3.Y3
Out[11]= {2 (1 - q) - 10 q}

In[21]:= X5 = {p, 1 - p}; B5 = (  2  -2 ); Y5 = ( 0 ); X5.B5.Y5
                                ( -1   1 )       ( 1 )
Out[21]= {1 - 3 p}

In[22]:= X5 = {p, 1 - p}; B5 = (  2  -2 ); Y5 = ( 1 ); X5.B5.Y5
                                ( -1   1 )       ( 0 )
Out[22]= {-1 + 3 p}
```

b. To calculate the expected payoff of the firms sufficient to substitute into expressions (4) and (5) found values p*=1/3 and q*=3/14. Thus we have H1=-1+2q*=-4/7≈-0.57; H2= -1+3p*=0.
As we see, an expected loss of the 1-st firm is $57000 and expected profit of the 2-nd firm is zero.

4. Conclusion

In this paper, we have looked at Nash equilibrium and I've learned how it can be specifically applied to our world. By looking at different instances and exercises, we have proved that in Nash equilibrium, the relative playoffs are always balanced. If given more time, I would like to further investigate about the ultimatum game because of its complexity. It has an infinite number of strategies per player and I would like to learn how the maximum benefit is calculated. Like the name suggests, game theory is in essence a theory which means that it is not perfect. So, another part I would like to learn more about is how accurate game theory actually is when applied to real-life scenarios and actual data.

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