Exploring Generating Functions and Combinatorial Identities

H. Koo

Northwood High School, Irvine, CA, United States

Abstract

Generating functions are algebraic objects that encode sequences as coefficients of power series, allowing problems about counting and recurrence relations to be translated into problems about algebraic manipulation. This paper introduces several major types of generating functions—ordinary, exponential, Fibonacci, and Dirichlet generating functions—and explores how each can be used to solve combinatorial and number-theoretic problems. Through examples involving counting, Pascal's triangle, the hockey-stick identity, Fibonacci numbers, the binomial theorem summation identity, and Euler's totient function, the paper shows how generating functions unify ideas from combinatorics, algebra, and number theory. These examples illustrate how generating functions turn recursive or case-based problems into systematic coefficient-extraction problems and provide elegant proofs of important identities.

I. Introduction to Generating Functions

First introduced by Abraham de Moivre in 1730, a generating function is a formal power series or polynomial whose coefficients encode information about a sequence of numbers. A generating function can be either infinite or finite, and both can be manipulated mathematically to derive new generating functions.

The information encoded within the coefficients of generating functions is often used for counting. The way this information behaves when generating functions interact makes them one of the most powerful tools in counting. We will solve a few example questions to concretize this concept of counting with generating functions.

1. Counting with Generating Functions

The first question will help us concretize how multiplying generating functions corresponding to different choices in a scenario gives the total number of possibilities.

Question 1

Two brothers, Andrew and Ben, walk into a pizza restaurant that offers three different toppings. The cheese slice costs $1, the pepperoni costs $2, and the shrimp costs $3. Andrew has no food allergies, but Ben is allergic to shrimp and has to choose between the cheese and pepperoni slice. If the brothers are full, they may not order at all.

Determine the number of ways Andrew and Ben could have ordered for each total of $5, $4, . . . , and $0.

Solution

Andrew can choose from three different slices or decide not to order at all, so his generating function would be $1 + x + x^2 + x^3$. The exponent of x denotes the price of each slice such that $1 \cdot x^3$ represents one way of ordering a $3 slice, and so on. Meanwhile, Ben only has two slices to choose from, so his generating function is $1 + x + x^2$.

We can now find the total number of ways by multiplying the two functions such that

$$(1 + x + x^2 + x^3)(1 + x + x^2) = 1 + 2x + 3x^2 + 3x^3 + 2x^4 + x^5.$$

Once again, the exponents of x denote the cost of the order, and its coefficients denote the number of ways that total can be achieved. Therefore, there would be one way to order a total of $5, two ways for $4, three ways for $3, three ways for $2, two ways for $1, and one way for $0.

For more complex multiplications, the binomial theorem can assist in finding the correct coefficients. The binomial theorem states that the coefficients of all terms in $(x + y)^n$ are equivalent to the numbers in row n of Pascal's triangle. For example, the coefficients of the expansion $(x + y)^4$—1, 4, 6, 4, and 1—are identical to the fourth row of Pascal's triangle. Keep in mind that the 1 at the top of Pascal's triangle corresponds to the 0th row and 0th entry.

According to Pascal's triangle, the kth entry of the nth line is (k^n). This means that we can replace $(1 + x)^n$'s coefficients with combinations found in the triangle, giving us the equation

$$(1 + x)^n = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \binom{n}{2} x^2 + \ldots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$$

We can now use the coefficients of this equation to count how many times an event may occur. For example, $\binom{n}{k}$, the coefficient of x^k, represents how many times k individuals can be selected from a group of size n. Now, let us use this connection between binomial coefficients and combinations to solve a more complex version of the pizza problem.

Question 2

Andrew returns to the pizza shop with twenty-nine classmates (including himself), but this time, he and his classmates are limited to the $1 and $2 slice. Meanwhile, Ben allows himself to order the $3, $5, or $9 slice. All twenty-nine in Andrew's class and Ben are hungry and will order a slice. If all thirty orders were on the same bill, how many ways could the total be $45?

Solution

First, let's create a generating function for Andrew's class of twenty-nine people. Since generating functions must be multiplied to find the total possibilities, Andrew's generating function is

$$(x + x^2)^{29}$$

Ben's generating function is

$$x^3 + x^5 + x^9.$$

The number of orders totaling $45 is the coefficient of x45 in the multiplication

$$(x + x^2)^{29} \cdot (x^3 + x^5 + x^9).$$

Since there are three terms in the expression for Ben, we may separate this into three cases. For the first case, let us multiply the coefficient of x^{42} in Andrew's function and the coefficient of x^3 in Ben's to find the coefficient of x^{45}. Similarly, we should find the coefficients of x^{40} and x^{36}, each corresponding to x^5 and x^9.

Due to the binomial theorem, we know that the coefficient of the kth term of $(x + x^2)^{29}$ is $\binom{29}{k}$. Therefore, we can express the expansion of $(x + x^2)^{29}$ as a summation:

$$\sum_{k=0}^{29} \binom{29}{k} x^{29+k}$$

which tells us that (29) is the coefficient of x29+k. This means the coefficients of x^{42}, x^{40}, and x^{36} are $\binom{29}{17}$, $\binom{29}{11}$, and $\binom{29}{7}$, respectively.

Adding $\binom{29}{17}$, $\binom{29}{11}$, and $\binom{29}{7}$, gives us a total of 88, 054, 005 different ways of ordering to a total of $45. This demonstrates how generating functions can provide a systematic way to solve complex counting problems with ease.

We can also use generating functions instead of stars and bars in some questions. Here is an example.

Question 3

For all nonnegative integers a, b, and c, $a + b + c = 20$. Find the number of sets $\{a, b, c\}$.

Solution 1

The stars and bars method states that the number of ways of splitting n indistinguishable balls into k distinguishable bins is $\binom{n+k-1}{k-1}$. Let 20 represent indistinguishable balls and a, b, and c

represent three distinguishable bins. Applying stars and bars, the answer is

$$\binom{20+3-1}{3-1} = \binom{22}{2} = 231.$$

Solution 2

Let us first assign integer a the generating function of $1 + x + x^2 + \cdots + x^{20}$ such that the coefficients of 1, x, and x^2 denote the ways a could be 0, 1, and 2. Then, let us assign the same function for all a, b, and c, multiplying the first few terms so that

$$1 + x + x^2 + \cdots + x^{20} = 1 + 3x + 6x^2 + 10x^3 + \ldots$$

From this, we notice that the coefficient of x^n is the sum of all positive integers not exceeding $n + 1$. For example, for the coefficient of $6x^3, 6 = 1 + 2 + 3, and\ 10x^3, 10 = 1 + 2 + 3 + 4$. This means that the coefficient of x^k would be $\frac{(k+1)(k+2)}{2}$, the expression to calculate the sum of positive integers less than or equal to $n + 1$.

The coefficient of $x^2 0$ would then be $\frac{(21)(22)}{2} = 231$, matching the answer of Solution 1.

Now that these three questions have demonstrated how generating functions can be used for counting, let us explore different properties of generating functions.

2. Pascal's Triangle and Generating Functions

Pascal's triangle and generating functions seem to have a connection. Let us explore this connection by calculating the first terms of these expansions.

$(1 + x + x^2 + \cdots)^2 = 1 + 2x + 3x^2 + 4x^3 + \ldots$

$(1 + x + x^2 + \cdots)^3 = 1 + 3x + 6x^2 + 10x^3 + \ldots$

$(1 + x + x^2 + \cdots)^4 = 1 + 4x + 10x^2 + 20x^3 + \ldots$

Let us replace all coefficients of all equations with combinations.

$(1 + x + x^2 + \cdots)^2 = \binom{1}{1} + \binom{2}{1}x + \binom{3}{1}x^2 + \binom{4}{1}x^3 + \ldots$

$(1 + x + x^2 + \cdots)^3 = \binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 + \binom{5}{2}x^3 + \ldots$

$(1 + x + x^2 + \cdots)^4 = \binom{3}{3} + \binom{4}{3}x + \binom{5}{3}x^2 + \binom{6}{3}x^3 + \ldots$

We know that $\binom{n}{k}$ is the kth entry of the nth line in Pascal's triangle. This means that these since k stays constant for all coefficients, the coefficients would match the line of numbers descending the triangle. Let us visualize this through the hockey-stick identity.

a. Hockey-stick Identity

Here is Pascal's triangle.

1

1 1

1 2. 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

The numbers inside a Pascal's triangle are determined by the sum of the two adjacent numbers that are above the number. Here is a hockey stick.

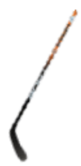

Figure 1. A Hockey Stick

Starting from any of the 1s on the edges of the triangle, move down the rows in a straight line, changing direction at the end.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

This forms a hockey-stick shaped figure, hence its name: the hockey-stick identity.

Note that the shaft, or the long section of the stick, resembles the coefficients from the equation

$(1 + x + x^2 + \cdots)^2 = \binom{1}{1} + \binom{2}{1}x + \binom{3}{1}x^2 + \binom{4}{1}x^3 + \ldots$

Meanwhile, the blade, or the short part of the stick, seems to denote the sum of all the coefficients.

Here is another example.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

The shaft of the yellow hockey stick in this example represents the coefficients of

$(1 + x + x^2 + \cdots)^3 = \binom{2}{2} + \binom{3}{2}\cdot x + \binom{4}{2}x^2 + \binom{5}{2}x^3 + \ldots$

and the 20 on the blade shows the sum of all the numbers on the shaft such that $20 = 1 + 3 + 6 + 10$. We will prove the hockey stick in two different ways. The first way is with combinations.

We want to determine the number of ways to choose a subset of three numbers from the set {1, 2, 3, 4, 5, 6}. The straightforward approach is to use $\binom{6}{3}$, which counts the number of ways to select three items from a set of six. This gives us 20 ways.

We can approach this problem in another way by considering multiple cases with where the smallest number in each set is fixed. For example, since each subset contains three elements, there are $\binom{5}{2}$ sets with 1 as the smallest number, $\binom{4}{2}$ sets with 2 as the smallest number, and so on. Repeat this until the smallest number of the set becomes 4 such that the set is {4, 5, 6}, and add the combinations:

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = 20,$$

giving us the same answer as method one. Combining the two approaches, we get the equation

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} = \binom{6}{3}.$$

These combinations match that of the previous yellow hockey stick. The identity can be generalized as such:

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

We will now explore another way of proving the hockey stick identity: stars and bars. First, let there be a situation where a_k represents a nonnegative integer such that

$$a_1 + a_2 + a_3 + \ldots + a_k \le n$$

Separate each case where n = 0, 1, 2, 3, . . . and use stars and bars to solve each case such that the cases where

$$a_1 + a_2 + a_3 + \ldots + a_k = 0$$

is represented by

$$\binom{0+k-1}{k-1} = \binom{k-1}{k-1}.$$

Similarly, the scenario where

$$a_1 + a_2 + a_3 + \ldots + a_k = 1$$

can be represented by

$$\binom{1+k-1}{k-1} = \binom{k}{k-1},$$

and such. Now, add the combinations of all cases such that we get

$$\binom{k-1}{k-1} + \binom{k}{k-1} + \binom{k+1}{k-1} + \ldots + \binom{n+k-1}{k-1}$$

Meanwhile, we may convert the original inequality of

$$a_1 + a_2 + a_3 + \ldots + a_k \leq n$$

into

$$a_1 + a_2 + a_3 + \ldots + a_k + a_{k+1} = n$$

if we apply logic that for all the cases of the inequality where $n < 20, a_k + 1$ may store the value of $20 - n$. Using stars and bars, we get

$$\binom{n+(k+1)-1}{(k-1)-1} = \binom{n-k}{k}.$$

Combine this with the solution of the original inequality such that

$$\binom{k-1}{k-1} + \binom{k}{k-1} + \binom{k+1}{k-1} + \ldots + \binom{n+k-1}{k-1} = \binom{n+k}{k}.$$

Replace $k - 1$ with r to simplify the equation:

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{r+n}{r} = \binom{r+n+1}{r+1}.$$

Once again, we have proven why the hockey stick identity works.

Now that we have identified a connection between Pascal's triangle and the coefficients of generating functions, we can solve a more complex question.

3. More Counting with Generating Functions

First, note that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

since the generating function doubles as a geometric series starting with 1 and with a ratio of x. The following question will expand on this concept.

Question 4

Five friends will share a large pizza with 100 slices. Two of them want no more than one slice, two others want an odd number of slices, and the last person wants any number of slices. How many ways could these friends share the pizza?

Solution

Again, we must write generating functions for each case. The expression is $(1 + x)^2$for the two who wanted zero or one slice,

$(x + x^3 + x^5 + \cdots)^2$ for the two who wanted an odd number of slices, and $(1 + x + x^2 + x^3 + \cdots)$ for the last person who wanted any number of slices. The exponents denote the number of slices such that $1 \cdot x^3$ represents one way of a person wanting three slices. Note we can rewrite the two latter expressions into infinite geometric series such that

$$(x + x^3 + x^5 + \cdots)^2 = \left(\frac{x}{1-x^2}\right)^2,$$

and

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}.$$

Multiplying all of the generating functions, we get

$$(1+x)^2 . \left(\frac{x}{1-x^2}\right)^2 . \frac{1}{1-x} =$$

Use previous knowledge of expanding $(1 + x + x^2 + x^3 + \ldots)^3$ by replacing the coefficients with combinations such that

$$(1 + x + x^2 + x^3 + \ldots)^3 = \binom{2}{2} + \binom{3}{2}x + \binom{4}{2}x^2 + \binom{5}{2}x^3 + \ldots$$

and convert it once again to summation notation such that

$$\sum_{n=0}^{\infty} \binom{n+2}{2} x^n,$$

revealing that the coefficient of x^{97} is $\binom{99}{2}$. Therefore, there are a total of $\binom{99}{2}$, 4950 ways of dividing the 100 slices. Now that we used generating functions for different counting problems, we will explore different types of generating functions and their characteristics.

II. Ordinary Generating Functions

Let $a(x)$ and $b(x)$ each be an ordinary generating function.

$$a(x) = a_0x^0 + a_1x^1 + a_1x^1 + a_2x^2 + \cdots \quad (1)$$

$$b(x) = b_0x^0 + b_1x^1 + b_1x^1 + b_2x^2 + \cdots \quad (2)$$

1. Adding Ordinary Generating Functions

Let $b(x)$ be the sum of these functions such that

$$c(x) = a(x) + b(x).$$

Using equations (1) and (2) to expand the ordinary generating functions, we would get

$$c(x) = (a_0 + b_0) + (a_0 + b_0)x + (a_0 + b_0)x^2 + \ldots$$

Noting that there is a pattern, convert the complex equation in to the simple summation of

$$c(x) = \sum_{n=0}^{\infty} (a_k + b_k)x^k.$$

Now, now that we have added ordinary generating functions, let's find out what happens if we multiply them.

2. Multiplying Ordinary Generating Functions

Let $d(x)$ be the product of the two ordinary generating functions.

$$d(x) = a(x)b(x)$$

Our goal is to simplify $d(x)$ with summation notation. Expand $a(x)$ and $b(x)$ using equations (1) and (2), but stop after finding a few terms.

$$d(x) = (a_0b_0) + (a_1b_0) + (a_0b_1)x + (a_0b_2) + (a_1b_1)x^2 + \ldots$$

Note the pattern in the coefficient of x^2 which resembles the terms of the expansion $(x + y)^2 = x^2y^0 + 2x^1y^1 + x^0y^2$. This means another a summation within the coefficient may help simplifying $d(x)$. Let us do just that by first isolating the coefficients:

$$d(x) = \sum_{k=0}^{\infty}(a_0b_k + a_1b_{k-1} + \ldots a_kb_0)x^k.$$

Then, convert the coefficient into a summation:

$$d(x) = \sum_{k=0}^{\infty}\left(\sum_{k=0}^{\infty} a_{k-i}b^i\right)x^k.$$

We will now explore some characteristics of $d(x)$.

Exploring $d(x)$.

First, let a_k , b_k ,, and d_k , denote the coefficient of x^k in $a(x)$, $b(x)$ and $d(x)$, respectively.

For all $i = 1, 2, 3, \ldots$, let $b_0 = 1$ and $b_1 = 0$. Then, $b(x) = 1$ because the only remaining term in $b(x)$ would be b_0x^0, or 1. Given that $b(x) = 1$, $d(x) = a(x)$ and $d_k = a_k$.

Similarly, let $b_m = 1$ and $b_j = 0$ for all nonnegative integers j be such that $j \neq m$. From the simplification of $d(x)$, we have learned that $d_k = \sum_{i=0}^{k} a_{k-1}\, b_i$. Since all $b_j = 0$, we must disregard all terms with b_j in the summation, which leaves us with $d_k = a_{k-m}b_m$. Since $b_m - 1$, $d_k = a_{k-m}$. Before moving onto Fibonacci generating functions, we will solve a simple question.

Question 1

Given that $d(x) = a(x)b(x)$, $a(x) = \sum_{k=0}^{\infty} a_k\, x_k$, and $d_k = \sum_{i=0}^{k} a_i$, find $b(x)$.

Solution

Note that d_k is the sum of $a(x)'s$ coefficients up to a_k. We must determine $d(x)$ from d_k such that

$$d(x) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{\infty} a_i\right)x^k$$

We know that if $b(x)$ had been an ordinary generating function, d_k would have been $\sum_{i=0}^{k} a_{k-1}\, b_i$, but $d_k = \sum_{i=0}^{k} a_i$ for this question. Therefore, considering that $\sum_{i=0}^{k} a_k$ equals $\sum_{i=0}^{k} a_i$,

$$\sum_{i=0}^{k} b_i = 1.$$

This means

$$b(x) = 1 + x + x^2 + \ldots,$$

which can be simplified to

$$b(x) = \frac{1}{1-x}.$$

This shows how $\frac{1}{1-x}$ acts as a cumulative summation operator in the multiplication $a(x) \cdot 1$

$1 - x = d(x)$, where $d(x)$'s coefficients denote the sums of $a(x)$'s coefficients. Now that we have reviewed ordinary generating functions, we will move on to explore the Fibonacci generating function.

3. Fibonacci Generating Functions

Let $F(x)$ be the generating function for the Fibonacci sequence:

$$F(x) = \sum_{k=0}^{\infty} F_k x^k$$

where

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, \ldots.$

Thus,

$$F(x) = 0 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \ldots$$

We now derive a closed expression for $F(x)$. Since the Fibonacci numbers satisfy the recurrence

$$F_{k+2} = F_{k+1} + F_k \qquad (k \geq 0),$$

multiply both sides by x^{k+2} and sum over all $k \geq 0$:

$$\sum_{k=0}^{\infty} F_{k+2} x^{k+2} = \sum_{k=0}^{\infty} F_{k+1} x^{k+2} + \sum_{k=0}^{\infty} F_k x^{k+2}.$$

Rewrite each sum in terms of $F(x)$:

$$F(x) - F_0 - F_1 x = x(F(x) - F_0) + x^2 F(x).$$

Substituting $F_0 = 0$ and $F_1 = 1$, we get

$$F(x) - x = xF(x) + x^2 F(x).$$

Now solve for $F(x)$:

$$F(x) - xF(x) - x^2 F(x) = x,$$

$$F(x)(1 - x - x^2) = x,$$

so

$$F(x) = \frac{x}{1 - x - x^2}. \qquad (3)$$

This rational expression for $F(x)$ allows us to prove identities involving Fibonacci numbers.

a. A Summation Identity for Fibonacci Numbers

We will now prove that

$$1 + \sum_{i=0}^{k-1} F_1 = F_{k+1}.$$

Equivalently,

$$\sum_{i=0}^{k} F_1 = F_{k+2} - 1$$

Let

$$G(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} F_1 \right) x^k.$$

Since multiplying by $\frac{1}{1-x}$ takes cumulative sums of coefficients,

$$G(x) = \frac{F(x)}{1 - x}.$$

Using equation (4), this becomes

$$G(x) = \frac{x}{(1 - x)(1 - x - x^2)}.$$

Now consider the generating function of the sequence $F_{k+2} - 1$:

$$\sum_{k=0}^{\infty} (F_{k+2} - 1)\, x^k = \sum_{k=0}^{\infty} F_{k+2} x^k - \sum_{k=0}^{\infty} x^k.$$

The first sum is

$$\sum_{k=0}^{\infty} F_{k+2} x^k = \frac{F(x) - x}{x^2},$$

since removing the first two terms of $F(x)$ and dividing by x^2 shifts the indices down by two. Therefore,

$$\sum_{k=0}^{\infty} (F_{k+2} - 1) x^k = \frac{F(x) - x}{x^2} - \frac{1}{1 - x}.$$

Substitute $F(x) = \frac{x}{1-x-x^2}$:

$$\frac{\frac{x}{1-x-x^2} - x}{x^2} - \frac{1}{1-x}$$
$$= \frac{1}{1-x-x^2} - \frac{1}{1-x}.$$

Combine the fractions:

$$\frac{1}{1-x-x^2} - \frac{1}{1-x} = \frac{x}{(1-x)(1-x-x^2}.$$

But this is exactly G(x). Since the two generating functions are equal, their coefficients are equal:

$$\sum_{i=0}^{k} F_i = F_{k+2} - 1.$$

Replacing k with $k - 1$ gives

$$1 + \sum_{i=0}^{k} F_i = F_{k+1}.$$

We now prove another theorem related to Fibonacci generating functions.

a. A Closed Formula for Fibonacci Numbers

A well-known identity state that

$$F_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k}.$$

The floor function appears because the value of k cannot be so large that $n-k-1 < k$. Equivalently, we must have

$$2k \leq n-1,$$

which is why the upper bound is $\left[\frac{n-1}{2}\right]$.

We will prove this identity by showing that the generating function of the right-hand side is equal to the Fibonacci generating function.

Define

$$A_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k}.$$

Now consider the generating function

$$A(x) = \sum_{n-1}^{\infty} A_n x^n = \sum_{n-1}^{\infty} \left(\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} \right) x^n.$$

We now re-index the inner expression. Let

$$m = n-k-1.$$

Then

$$n = m-k-1.$$

and the condition $k \leq \left[\frac{n-1}{2}\right]$ becomes $m \geq k$. Therefore,

$$A(x) = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \binom{m}{k} x^{m+k+1}.$$

Factor out powers of x:

$$A(x) = x \sum_{k=0}^{\infty} x^k \sum_{m=k}^{\infty} \binom{m}{k} x^m.$$

Now use the standard generating-function identity

$$\sum_{m=k}^{\infty} \binom{m}{k} x^m = \frac{x^k}{(1-x)^{k+1}}.$$

Substituting this into $A(x)$, we get

$$A(x) = x \sum_{k=0}^{\infty} x^k \cdot \frac{x^k}{(1-x)^{k+1}}.$$

Simplifying,

$$A(x) = \frac{x}{1-x} \sum_{k=0}^{\infty} \left(\frac{x^2}{1-x} \right)^k.$$

The remaining sum is geometric,

$$A(x) = \frac{x}{1-x} \cdot \frac{1}{1-\frac{x^2}{1-x}}.$$

Simplify the denominator:

$$A(x) = \frac{x}{1-x-x^2}.$$

But equation (4) showed that

$$F(x) = \frac{x}{1-x-x^2}.$$

Thus,

$$A(x) = F(x).$$

Since equal generating functions have equal coefficients, it follows that

$$A_n = F_n.$$

and therefore

$$F_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k}.$$

This identity gives a closed formula for the Fibonacci numbers in terms of binomial coefficients. It shows that the Fibonacci sequence, which is usually defined recursively, can also be expressed directly through combinations.

b. A Closed Formula for Fibonacci Numbers

A well-known theorem states that the Fibonacci numbers can be written as

$$F_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k}.$$

We will prove this by showing that the generating function of the right-hand side is the Fibonacci generating function.

Let

$$A_n = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k}.$$

We will compute the ordinary generating function

$$A(x) = \sum_{n-1}^{\infty} \left(\sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-k-1}{k} \right) x^n$$

Now re-index by letting $m = n - k - 1$. Then $n = m + k + 1$, and the condition $k \leq [(n-1)/2]$ becomes $m \geq k$. Therefore,

$$A(x) = \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \binom{m}{k} x^{m+k+1}$$

Factor out x and separate powers:

$$A(x) = x \sum_{k=0}^{\infty} x^k \sum_{m=k}^{\infty} \binom{m}{k} x^m.$$

We now use the identity

$$\sum_{m=k}^{\infty} \binom{m}{k} x^m = \frac{x^k}{(1-x)^{k+1}}.$$

Substituting this into the generating function gives

$$A(x) = x \sum_{k=0}^{\infty} x^k \cdot \frac{x^k}{(1-x)^{k+1}}.$$

Simplify:

$$A(x) = \frac{x}{1-x} \sum_{k=0}^{\infty} \left(\frac{x^2}{1-x} \right)^k.$$

The remaining sum is geometric, so

$$A(x) = \frac{x}{1-x} \cdot \frac{1}{1-\frac{x^2}{1-x}}.$$

Simplifying the denominator,

$$A(x) = \frac{x}{1 - x - x^2}.$$

But this is exactly the Fibonacci generating function $F(x)$. Therefore, the coefficients must be equal, so

$$A_n = F_n.$$

Hence,

$$F_n = \sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \binom{n-k-1}{k}.$$

This identity shows that F_n counts the number of ways to choose k non-adjacent positions among $n - 1$ available slots, summed over all possible values of k. It is another example of how generating functions turn recurrence relations into explicit formulas.

4. Exponential Generating Functions

Exponential generating functions can be defined as an ordinary generating function with each of its term divided by the factorial of such exponent.

$$g(x) = g_0\frac{x^0}{0!} + g_1\frac{x^1}{1!} + g_2\frac{x^2}{2!} + \ldots$$

Simplified into a summation, this is

$$g(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}.$$

Now, we will explore some properties of exponential generating functions and utilize them to prove a theorem.

a. Multiplying Exponential Generating Functions

To multiply two exponential generating functions, let there be another exponential generating function

$$h(x) = h_0\frac{x^0}{0!} + h_1\frac{x^1}{1!} + h_2\frac{x^2}{2!} + \ldots$$

Multiply these functions and call it $d(x)$ such that

$$d(x) = g(x) \cdot h(x),$$

and

$$d(x) = \left(g(x) = g_0\frac{x^0}{0!} + g_1\frac{x^1}{1!} + g_2\frac{x^2}{2!} + \ldots\right)\left(h(x) = h_0\frac{x^0}{0!} + h_1\frac{x^1}{1!} + h_2\frac{x^2}{2!} + \ldots\right)$$

Now, expand this such that we get the first few terms

$$d(x) = \frac{g_0h_0}{0!0!}x^0 + \left(\frac{g_1h_0}{1!0!} + \frac{g_0h_1}{0!1!}\right) + \left(\frac{g_2h_0}{2!0!} + \frac{g_1h_1}{1!1!} + \frac{g_0h_2}{0!2!}\right) + \ldots$$

Note that both the numerator and denominator have numbers that can be found in a binomial expansion. Simplify such coefficients so that for d_m denoting the coefficient of x^m,

$$d_m = \sum_{n=0}^{m} \frac{g_m - nh_n}{(m-n)n!}.$$

Input this summation into the generating function $d(x)$ such that

$$d(x) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \frac{g_m - nh_n}{(m-n)n!}\right)x^m.$$

Let $d(x)$ resemble an exponential generating function by adding $m!$ as a denominator of x^m, and multiply it to x^m's coefficient to neutralize the process such that

$$d(x) = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \frac{m! \cdot g_m - nh_n}{(m-n)n!} \right) \frac{x^m}{m!}$$

Note that there is the generalized expression for $\binom{m}{n}$ inside the summation, and utilize this information by isolating it such that

$$d(x) = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \frac{m!}{(m-n)n!} \cdot g_m - nh_n \right) \frac{x^m}{m!}$$
$$= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} \binom{m}{n} \cdot g_m - nh_n \right) \frac{x^m}{m!}$$

Now that we have established how to multiply exponential generating functions, we will apply these techniques to prove an important combinatorial identity.

5. The Binomial Theorem Summation Identity

The binomial theorem summation identity states that

$$\sum_{r=0}^{n} \binom{n}{r} = 2".$$

We will prove this with three different methods: combinations, induction, and exponential generating functions.

a. Proof by Combinations

In order to use counting, create an example question with a similar premise.

Question 1

How many ways can Andrew order a pizza if he started with a cheese pizza and could add pepperoni, olives, and sausages?

Solution 1

Use $\binom{3}{0}$ to represent the cheese pizza without any toppings, $\binom{3}{1}$ to represent choosing one topping, $\binom{3}{2}$ to represent two toppings, and $\binom{3}{3}$ to represent choosing all three toppings for the pizza. Altogether, this makes

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8.$$

This alone does not yet prove the summation identity, so we will solve the question another way.

Solution 2

Since each topping has two cases of either being on or of the pizza, we can represent all cases such that

$$2 \cdot 2 \cdot 2 = 2^3.$$

If we generalize this example question such that there are n toppings available, the first solution would be $\sum_r^n \binom{n}{r}$, and the second solution would be $2n$. Therefore,

$$\sum_{r=0}^{n} \binom{n}{r} = 2",$$

proving the binomial theorem summation identity. Now that we have used combinations to prove the identity, we will use induction to prove it.

b. Proof by Induction

We will now prove the binomial theorem summation identity using mathematical induction. First, check the base case where $n =$

0. The summation simplifies to

$$\sum_{r=0}^{0} \binom{0}{r} = \binom{0}{0}$$

$$\sum_{r=0}^{0} \binom{0}{r} = 1.$$

Since $2^0 = 1$, the base case holds. Assume the identity holds for some integer $n = k$, such that

$$\sum_{r=0}^{k} \binom{k}{r} = 2^k.$$

Then we will try the case n = k + 1 to fully generalize this identity and prove that it works for all n.

For the case of $n = k + 1$, expand the sum using Pascal's identity such that

$$\sum_{r=0}^{k+1} \binom{k+1}{r} = \sum_{r=0}^{k+1} \left(\binom{k}{r} + \binom{k}{r-1} \right).$$

Splitting this into two separate sums,

$$\sum_{r=0}^{k+1} \binom{k+1}{r} = \sum_{r=0}^{k} \binom{k}{r} + \sum_{r=1}^{k+1} \binom{k}{r-1}.$$

Rewriting the second summation with an index shift $(r' = r - 1)$,

$$\sum_{r=0}^{k+1} \binom{k}{r-1} = \sum_{r'=0}^{k} \binom{k}{r'}$$

Since both sums are now identical,

$$\sum_{r=0}^{k+1} \binom{k+1}{r} = \sum_{r=0}^{k} \binom{k}{r} + \sum_{r=0}^{k} \binom{k}{r}.$$

By the inductive hypothesis,

$$\sum_{r=0}^{k} \binom{k}{r} = 2^k.$$

Thus,

$$\sum_{r=0}^{k+1} \binom{k+1}{r} = 2^k + 2^k = 2^{k+1},$$

proving that the theorem works for all cases. Since the base case is valid and we have shown that the formula holds for $n = k + 1$ whenever it holds for $n = k$, by the principle of mathematical induction, the identity

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

holds for all non-negative integers n.

Now that we have proved the identity using induction, we will use exponential generating functions as our final approach.

c. Proof by Exponential Generating Functions

The Maclaurin series for ex is as follows:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We will prove the binomial theorem summation identity by expressing e^{2x} in two different ways. The first way is such that

$$e^{2x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

and the second way is such that

$$e^{2x} = e^x \cdot e^x = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left(\sum_{m=0}^{\infty} \frac{x^m}{m!} \right)$$

Then we find the coefficient of x^n in each equation such that by

$$e^{2x} = \left(\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}\right) = \left(\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}\right)$$

we get

$$\frac{2^n}{n!}$$

For the second equation, we expand it such that

$$e^{2x} = e^x \cdot e^x = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{m=0}^{\infty} \frac{x^m}{m!}\right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{n=0}^{n} \frac{1}{k!\,(n-k)!}\right)$$

and get

$$\sum_{n=0}^{n} \frac{1}{k!\,(n-k)!}$$

as the coefficient of x^n. Setting these two coefficients as equal, we get

$$\frac{2^n}{n!} = \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!}$$

Multiply both sides with $n!$ to get

$$2^n = \sum_{k=0}^{n} \frac{1}{k!\,(n-k)!}$$

Then, after converting the fraction inside the summation to a combination, we get

$$2^n = \sum_{k=0}^{n} \binom{n}{k},$$

once again proving the binomial theorem summation identity.

6. Dirichlet Generating Functions

A Dirichlet generating function can be defined as such:

$$a(s) = \sum_{n-1}^{\infty} a_n \frac{1}{n^s} = a_1 \frac{1}{1^s} + a_2 \frac{1}{2^s} + a_3 \frac{1}{3^s} + \ldots.$$

a. Multiplying Dirichlet Generating Functions

To find out what happens when we multiply two Dirichlet generating functions, let

$$b(s) = \sum_{n-1}^{\infty} b_n \frac{1}{n^s}.$$

Then let $d(s) = a(s) \cdot b(s)$ and expand such that, $d(s) = a(s) \cdot b(s)$

$$= \left(\sum_{n-1}^{\infty} a_n \frac{1}{n^s}\right)\left(\sum_{n-1}^{\infty} b_n \frac{1}{n^s}\right)$$

$$= \left(a_1 \frac{1}{1^s} + a_2 \frac{1}{2^s} + a_3 \frac{1}{3^s} + \ldots\right)\left(b_1 \frac{1}{1^s} + b_2 \frac{1}{2^s} + b_3 \frac{1}{3^s} + \ldots\right)$$

$$= a_1 b_1 \frac{1}{1^s} + (a_1 b_2 + a_2 b_1)\frac{1}{2^s} + (a_1 b_3 + a_3 b_1)\frac{1}{3^s} + (a_1 b_4 + a_2 b_2 + a_4 b_1)\frac{1}{4^s} + (a_1 b_5 + a_5 b_5)\frac{1}{5^s} + (a_1 b_6 + a_2 b_3 + a_3 b_2 + a_6 b_1)\frac{1}{6^s} + \ldots$$

This expansion reveals a pattern in the coefficients of the fractions, in which each ab pair represents a pair of factors of the number in the denominator of the fraction. We can use this fact to simplify $d(s)$ into a summation such that

$$d(s) = \sum_{n-1}^{\infty} \left(\sum_{d \backslash n} a_d\, b_{n/d}\right)\frac{1}{n^s}.$$

b. Riemann Zeta Generating Function

A particularly important example of a Dirichlet generating function is obtained when every coefficient is equal to 1. This gives the Riemann zeta function,

$$\zeta(s) = \sum_{n-1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots$$

Multiplying two copies of ω(s) illustrates how Dirichlet generating functions encode divisor structure:

$$\zeta(s)^2 = 1\frac{1}{1^s} + 2\frac{1}{2^s} + 2\frac{1}{3^s} + 3\frac{1}{4^s} + 2\frac{1}{5^s} + 4\frac{1}{6^s} + \ldots$$

The coefficient of $\frac{1}{n^s}$ counts the number of positive divisors of n. For example, 6 has four positive divisors —1, 2, 3, and 6—so the coefficient of $\frac{1}{6^s}$ is 4.

This example shows that Dirichlet generating functions are especially useful when a sequence is defined by divisibility properties rather than by addition of indices, as in ordinary generating functions. We now turn to another important arithmetic function: Euler's totient function.

c. Totient Function

Another important arithmetic function is Euler's totient function, denoted by $\phi(n)$. It counts the number of integers from 1 to n that are relatively prime to n. For example,

$$\phi(1) = 1,\ \phi(2) = 1,\ \phi(3) = 2,\ \phi(4) = 2,$$

$$\phi(5) = 4, \phi(6) = 2, \phi(7) = 6, \phi(8) = 4.$$

The value of $\phi(n)$ depends on the prime factorization of n. If n is prime, then every positive integer less than n is relatively prime to n, so

$$\phi(n) = \mathrm{n} - 1.$$

More generally, if $n = p^e$ for a prime p, then exactly one out of every p integers from 1 to n is divisible by p. Therefore,

$$\phi(n) = \mathrm{n}\left(1 - \frac{1}{p}\right).$$

If n has prime factorization

$$n = p_1^{e1} p_2^{e2} p_3^{e3} \ldots,$$

then the same reasoning extends multiplicatively:

$$\phi(n) = \mathrm{n}\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)\ldots$$

For example, since

$$100 = 2^2 \cdot 5^2,$$

we get

$$\phi(100) = 100\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{5}\right) = 100 \cdot \frac{1}{2} \cdot \frac{4}{5} = 40.$$

Because $\phi(n)$ is defined in terms of divisibility and prime factorization, it is especially natural to study it using Dirichlet generating functions.

Just as ordinary generating functions help study sequences defined by additive structure, Dirichlet generating functions help study arithmetic functions defined by divisor structure. This makes them a natural tool for analyzing Euler's totient function.

d. Dirichlet Generating Function of the Totient Function

The Euler totient function $\phi(n)$ counts the number of integers from 1 to n that are relatively prime to n. A fundamental identity involving

$\phi(n)$ is

$$\sum_{d\backslash n} \phi(d) = n.$$

This identity states that if we sum the totients of all positive divisors of n, we get n.

We can rewrite this using Dirichlet convolution. Let 1(n)=1 for all n, and let $id(n) = n$. Then the identity above becomes

$$\phi * 1 = id,$$

where * denotes Dirichlet convolution.

Now use Dirichlet generating functions. The Dirichlet generating function of $1(n)$ is

$$\sum_{n-1}^{\infty} \frac{1}{n^s} = \zeta(s),$$

and the Dirichlet generating function of $id(n) = n$ is

$$\sum_{n-1}^{\infty} \frac{n}{n^s} = \sum_{n-1}^{\infty} \frac{n1}{n^{s-}} = \zeta(s-1).$$

Since multiplication of Dirichlet generating functions corresponds to Dirichlet convolution, we get

$$\left(\sum_{n-1}^{\infty} \frac{\phi(n)}{n^s}\right)\zeta(s) = \zeta(s-1).$$

Dividing both sides by $\zeta(s)$, we obtain

$$\sum_{n-1}^{\infty} \frac{\phi(n)}{n^s} = \frac{\zeta(s-1)}{\zeta(s)}.$$

This identity is important because it connects an arithmetic function, $\phi(n)$, to the Riemann zeta function. It shows that Dirichlet generating functions are not only formal tools for manipulating sequences, but also a bridge between combinatorics and number theory.

Conclusion

Generating functions translate sequences into algebraic objects that can be added, multiplied, and transformed. In counting problems, they encode choices and make coefficient extraction a systematic way to count possibilities. In recurrence relations such as the Fibonacci sequence, they turn recursive definitions into closed-form functional equations. In number theory, Dirichlet generating functions connect arithmetic functions such as $\phi(n)$ to the Riemann zeta function. Across these settings, generating functions provide a unified language for solving problems in combinatorics, algebra, and number theory.

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