Analysis of relationship between polynomial and complex numbers

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Abstract

The paper introduces different forms of representing complex numbers; standard form and polar form. Each has its own pros and cons, and each can be applied in different situations. The paper focuses on polar form and shows how it can be applied for finding out factor-square property(FSP) polynomials. Modular arithmetic is also required for finding FSP polynomials so that the paper explains about basics of modular arithmetic as well.

Standard Form

The numbers that people are most familiar with are real numbers. Based on the characteristics of real numbers, it is known that negative numbers don't have real square roots as a square is either positive or 0. However, mathematicians created the symbol i to represent a number whose square is -1. An imaginary number is defined as a real number multiplied by the imaginary unit i.

$i = \sqrt{-1}$

A complex number in standard form has two parts: the real part and the imaginary part. Every complex number can be expressed as the form $a + bi$, where $a, b \in \mathbb{R}$. We often use z as a variable to represent complex numbers, rather than x or y that are typically used to express real numbers.

$z = a + bi$

In real numbers, a conjugate is created by altering the sign of two binomial expressions. The conjugate of $x + y$, for example, is $x - y$. The two binomials are conjugates of each other. This is similar for complex numbers. Let $z = a + bi$. Then, $a - bi$ is the conjugate of z, referred as z.

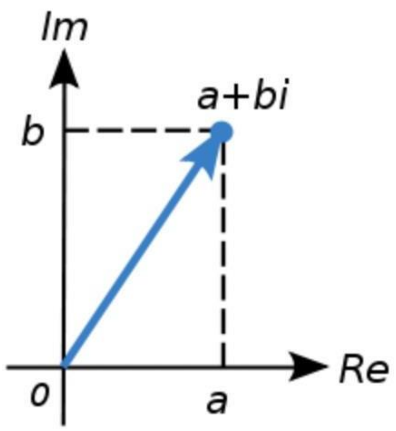
$z = a - bi$

A complex number can be plotted on the

complex plane where the horizontal axis is for the real part, and the vertical axis is for the imaginary part. The complex plane acts very similar to the Cartesian plane. Plotting the complex number $a + bi$ is the same as plotting (a, b) in the Cartesian plane. The horizontal axis is often labeled as Re and the vertical axis is often labeled as Im.

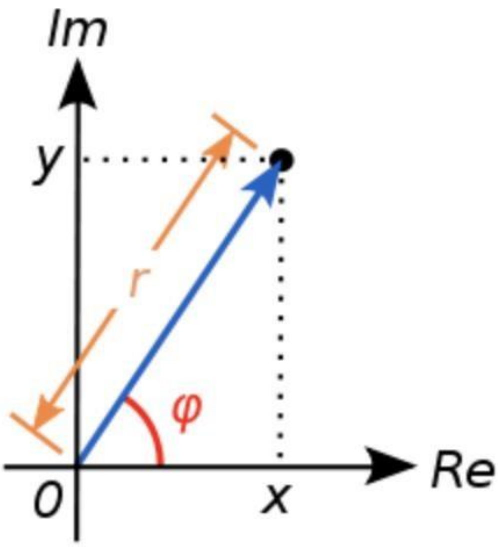
For a complex number $z = a + bi$, the absolute value of z can be represented as $|z|$, This expression is defined as the distance from z to the origin in the complex plane. We can find the value by using the Pythagorean Theorem. Referring to the diagram below, we can know that $|z|$

$= \sqrt{a2 + b2}.$



Polar Form

The most common method to plot a point on the Cartesian plane is to use x-coordinates and y- coordinates. However, this is not the only method. Polar coordinates use direction and distance to identify the location of a point in the Cartesian plane. Polar coordinates consist of two coordinates: the r-coordinate and the θ-coordinate. The r- coordinate is the distance from the origin to the point, while the θ -coordinate is used the same way in trigonometry. For example, a point with $\theta = 0°$ is directly to the right of the origin, and a point with $\theta = 90°$ is directly above the origin.



Polar coordinates are used to create a polar form of a complex number. Polar form is another way to represent complex numbers.

Conversion between Standard form and Polar form

Every complex number in standard form, $a + bi$, can be written in the polar form $r(cos\ \theta + i$ sin $\theta)$, where $r \geq 0$. In $z = a + bi$. By using the Pythagorean's Theorem, we can know that the distance from the point z to the origin, r, is equal to $\sqrt{a2 + b2}$. By using the trigonometric ratios, we can know that $cos\ \theta = a/r$ and $sin\ \theta = b/\mathrm{r}$, where θ is the angle indicated in the diagram above. To get the value of θ, you need to solve

$tan{-}1(b/a)$, as tan θ =b . By a multiplying the two equations by r, we get $r\ cos\ \theta$ = a and $r\ sin\ \theta$ = b. Since z = a + bi, z =$rcos\ \theta$ + $irsin\ \theta$. This is why every complex number can be written in the form $r(cos\ \theta$ + i sin $\theta)$. The quantity $cos\ \theta$ + i sin θ is often abbreviated as $cis\ \theta$. Here is one example showing the conversion from standard form to polar form of a complex number:

Let z = 1 + i. z in polar form would be

$$\sqrt{2}\,(cos\frac{\pi}{4} + isin\frac{\pi}{4}) \text{ as } r = \sqrt{1^2 + 1^2} = \sqrt{2} \text{ and } \theta = tan^{-1}(\frac{1}{1}) = \frac{\pi}{4}$$

To convert polar form to standard form, just evaluate trigonometric values simplify. One example of showing the conversion from polar from to standard form of a complex number:

$$\text{Let } z = 3\,(cos\frac{7\pi}{6} + isin\frac{7\pi}{6}).$$

z in polar form would be

$$-\frac{3\sqrt{3}}{2} - \frac{3}{2}i \text{ as } cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$

and $sin\ 7\pi/6$ = -1/2

Addition and Subtraction of Complex Numbers

Just like addition and subtraction between real numbers, we can perform arithmetic operations on complex numbers. Let z = a + bi and w = c + di. Here is the process of adding and subtracting these two complex numbers.

$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$

$z - w = (a + bi) - (c + di) = (a - c) + (b - d)i$

Let z = 3 + 4i, w = 2 + i. Here is one example for addition and subtraction using standard form:

$z + w = (3 + 4i) + (2 + i) = (3 + 2) + (4 + 1)i = 5 + 5i$

$z - w = (3 + 4i) - (2 + i) = (3 - 2) + (4 - 1)i = 1 + 3i$

$$\text{Let } z = 2\,(cos\frac{\pi}{4} + isin\frac{\pi}{4}),\ w = (cos\frac{\pi}{6} + isin\frac{\pi}{6}).$$

Here is one example for addition and subtraction using polar form:

$$z + w = 2\,(cos\frac{\pi}{4} + isin\frac{\pi}{4}) + (cos\frac{\pi}{6} + isin\frac{\pi}{6}) = (2cos\frac{\pi}{4} + cos\frac{\pi}{6}) + (2sin\frac{\pi}{4} + sin\frac{\pi}{6})\,i$$
$$= (\sqrt{2} + \frac{\sqrt{3}}{2}) + (\sqrt{2} + \frac{1}{2})i$$
$$z - w = 2\,(cos\frac{\pi}{4} + isin\frac{\pi}{4}) - (cos\frac{\pi}{6} + isin\frac{\pi}{6}) = (2cos\frac{\pi}{4} - cos\frac{\pi}{6}) + (2sin\frac{\pi}{4} - sin\frac{\pi}{6})\,i$$
$$= (\sqrt{2} - \frac{\sqrt{3}}{2}) + (\sqrt{2} - \frac{1}{2})i$$

There are advantages and disadvantages for using standard form and polar form of complex numbers, respectively. Addition and subtraction between complex numbers is better using standard form because you just need to simply add the real parts added together to form the real part of the sum, and the imaginary parts to form the imaginary part of the sum. On the other hand, addition and subtraction using polar form is harder because trigonometric values need to be calculated.

Multiplication and Division of Complex Numbers

So why do we use polar form when you are more familiar with representing complex numbers in standard form? One reason is that it is easy to multiply and divide complex numbers with each other. Let z_1 = r_1($cos\ \theta_1$ + $i\ sin\ \theta_1$) = (a + bi) and z_2 = r_2($cos\ \theta_2$ + $i\ sin\ \theta_2$) = (c + di). Here is the process of multiplying and dividing these two complex numbers.

$$z_1z_2 = r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1))$$
$$= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$
$$== r_1r_2 cis(\theta_1+\theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$

$$= \frac{r_1}{r_2} cis(\theta_1 - \theta_2)$$

Let $w_1 = 6\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, $w_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

Here is one example for multiplication and division using polar form:

$$w_1w_2 = 6 \times 2\ cis\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = 12\ cis\left(\frac{5\pi}{12}\right)$$

$$\frac{w_1}{w_2} = \frac{6}{2} \times cis\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = 3\ cis\left(\frac{\pi}{12}\right)$$

Let $z = a + bi$, $w = c + di$. Here is the process of multiplying and dividing these two complex numbers in standard form.

$$z_1z_2 = (a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$$

$$\frac{z_1}{z_2} = \frac{(a+bi)}{(c+di)} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac - adi + bci + bd)}{c^2+d^2} = \frac{(ac+bd) + (bc-ad)i}{c^2+d^2}$$
$$= \frac{(ac+bd)}{(c^2+d^2)} + \frac{(bc-ad)}{(c^2+d^2)}i$$

Let $w1 = 5 + 7i$, $w2 = 8 - 3i$. Here is one example for multiplication and division using standard form.

$$w_1w_2 = (5+7i)(8-3i)$$
$$= 40 - 15i + 56i - 21i^2$$
$$= 61 + 41i$$

$$\frac{w_1}{w_2} = \frac{5+7i}{8-3i} = \frac{(5+7i)(8+3i)}{(8-3i)(8+3i)}$$

$$= \frac{40 + 15i + 56i + 21i^2}{64+9}$$

$$= \frac{19+71i}{73} = \frac{19}{73} + \frac{71}{73}i$$

Multiplication of complex numbers in polar form is equivalent to multiplying their magnitudes and adding their angles, while division of complex numbers in polar form is equivalent to dividing their magnitudes and subtracting their angles. On the other hand, multiplying and dividing using standard form is harder because we need to perform complex algebra to expand and simplify expressions. Therefore, using polar form when multiplying and dividing complex numbers is a more efficient method than using standard form.

De Moivre's Theorem with Inductive Proof

When is polar form of a complex number useful? The polar form is especially useful when we're working with powers and roots of a complex number. A theorem called "De Moivre's Theorem" facilitates the process of finding the powers and roots of complex numbers in complex form.

This is De Moivre's Theorem:

$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

To prove that this equation is true for any positive integer n, we need to use induction. Induction in mathematical terms is a technique used to prove that a statement is true for every natural number. Let n be a natural number that is the variable for the statement you are trying to prove.

Step 1. Show that the statement works when n = 1. Step 2. Assume that the statement works when n = k.

Step 3. Using the hypothesis that is set up for n = k, show that the statement works for n = k + 1.

This process resembles the "Domino Effect", with step 1 being the first domino and step 2 & 3 being the next dominos falling.

When n = 1: $(r(\cos\theta + i \sin\theta))1 = r(\cos\theta + i \sin\theta)$

Through the calculations above, we figured out that $z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) +$

$isin(\theta_1 + \theta_2))$.

Therefore, when n = 2: $(r(\cos\theta + i \sin\theta))^2 = r_2(\cos\theta + i \sin\theta)(\cos\theta + i \sin\theta) =$

$r_2(\cos(\theta + \theta) + \text{isin}(\theta + \theta) = r_2(\cos 2\theta + \text{isin } 2\theta)$

Assuming that n = k: $(r(\cos\theta + i \sin\theta))k = rk(\cos k\theta + i \sin k\theta)$,

When n = k+1: $(r(\cos\theta + i \sin\theta))k+1 = r(\cos\theta + i \sin\theta) \times rk(\cos\theta + i \sin\theta)k =$

$rk+1((\cos\theta + i \sin\theta)(\cos k\theta + i \sin k\theta)) = rk+1(\cos(\theta + k\theta) + \text{isin}(\theta + k\theta)) =$

$rk+1(\cos((k + 1)\theta) + \text{isin}((k + 1)\theta))$

Therefore, $(r(\cos\theta + i \sin\theta))n = rn(\cos n\theta + i \sin n\theta)$ for all positive integers n.

Nth Roots

The Fundamental Theorem of Algebra states that that every polynomial with degree n has n roots. The method of finding the nth root helps us find the roots of a polynomial, either real or non-real. The equation below is one of the best equations to display the method of finding nth roots. The number n is a positive integer.

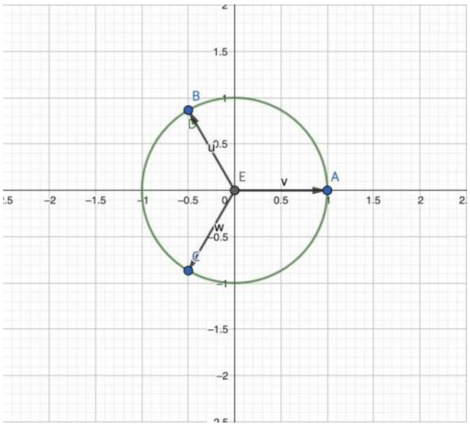
$f(x) = xn - 1$

When $n = 3$,

$f(x) = x^3 - 1$

If we factor $f(x)$, we get $f(x) = (x - 1)(x2 + x + 1)$. Knowing only one root, we can find the other two roots using the quadratic formula. The three roots are 1, $(-1+\sqrt{3i})/2$, $(-1-\sqrt{3i.})/2$ Transforming these roots into polar form gives us $(cos0 + isin0)$, $(cos\frac{2\pi}{3} + isin\ \frac{2\pi}{3})$, $(cos\ \frac{4\pi}{3} + isin\ \frac{4\pi}{3})$.

Mapping these roots in the complex plane, we get:



Here, we can know that the roots, when mapped in a complex plane, are evenly spaced out. Why are they evenly spread out? We should look at another example of finding nth roots. Let $x = r * cis\ \theta$.

$$x^4 = -8 - 8\sqrt{3}i$$
$$r^4 cis4\theta = 8(-1 - \sqrt{3}i)$$
$$r^4 cis4\theta = 8(2cis\frac{4\pi}{3})$$
$$r^4 cis4\theta = 16cis\frac{4\pi}{3}$$

From this, we get two equations:

1. $r^4 = 16$
2. $4\theta = \frac{4\pi}{3} + 2\pi n$, where $n \in \mathbb{Z}$

From equation 1, we can know that r = 2, as r represents a modulus(magnitude). From equation 2, we get $\theta = \pi/3 + \pi n/2$. Since we know that the total number of roots is four through the Fundamental Theorem of Algebra, the four roots would be $rcis\ \theta = rcis\ \pi/3 + \pi n/2$. where n is 0, 1, 2, and 3 respectively. The four roots are

$$2cis\frac{\pi}{3},\ 2cis\left(\frac{\pi}{3} + \frac{\pi}{2}\right),\ 2cis\left(\frac{\pi}{3} + \pi\right),\ 2cis\left(\frac{\pi}{3} + \frac{3\pi}{2}\right).$$

Since the angle of where the roots are mapped in a complex plane increase at a constant rate, the roots are evenly spaced when placed in a complex plane.

Let $w^n = z$, where $w, z \in \mathbb{C}$, $n \in \mathbb{Z}+$

Then, $w^n = r(\cos\theta + i\sin\theta)$, where $\theta \in \mathbb{R}$, $r \geq 0$.

When De Moivre's Theorem is used "backwards", then we can derive that:

$$w_k = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right),$$

where $k = 0, 1, 2, \cdots, n - 1$

These roots have the same magnitude r, but differ in their arguments by $2\pi/n$. This means that they are equally spaced on the circle with radius r centered at the origin.

There are n complex numbers that are nth roots of a given complex number. Every root has the same modulus r, but has n different arguments. Therefore, they are equally spaced complex numbers on the circle starting with the original argument divided by n.

Let $w^n = 32 - 32\sqrt{3}i = 64cis\frac{5\pi}{3}$.

$$z_0 = 64^{\frac{1}{6}}cis\left(\frac{\frac{5\pi}{3}}{6}\right) = 2cis\frac{5\pi}{18}.$$

$$z_1 = 64^{\frac{1}{6}}cis\left(\frac{\frac{5\pi}{3}}{6} + \frac{2\pi}{6}\right) = 2cis\frac{11\pi}{18}.$$

$$z_2 = 64^{\frac{1}{6}}cis\left(\frac{\frac{5\pi}{3}}{6} + \frac{4\pi}{6}\right) = 2cis\frac{17\pi}{18}.$$

$$z_3 = 64^{\frac{1}{6}}cis\left(\frac{\frac{5\pi}{3}}{6} + \frac{6\pi}{6}\right) = 2cis\frac{23\pi}{18}.$$

$$z_4 = 64^{\frac{1}{6}}cis\left(\frac{\frac{5\pi}{3}}{6} + \frac{8\pi}{6}\right) = 2cis\frac{29\pi}{18}.$$

$$z_5 = 64^{\frac{1}{6}}cis\left(\frac{\frac{5\pi}{3}}{6} + \frac{10\pi}{6}\right) = 2cis\frac{35\pi}{18}.$$

Number theory

Modular arithmetic is a branch of mathematics that deals with numbers and their remainders. It introduces the concept of a modulus, which represents the divisor used in the division process. The notation below shows that a and b have the same remainder when divided by modulus m.

$a \equiv b(mod\ m)$

One key property of modular arithmetic is that we can perform various operations, such as addition, subtraction, multiplication, and exponentiation. The properties below are examples of uses in modular arithmetic.

Given that $a \equiv b(mod\ m)$ and $c \in \mathbb{Z}+$,

1. $a + c \equiv b + c(mod\ m)$
2. $a - c \equiv b - c(mod\ m)$
3. $a \times c \equiv b \times c(mod\ m)$

4. $ac \equiv b^c(mod\ m)$

A polynomial $f(x)$ has the *factor-square property* (or FSP) if $f(x)$ is a factor of $f(x^2)$. For instance, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not.

Reason: $x - 1$ is a factor of $x^2 - 1$, and x is a factor of x^2, but $x + 2$ is not a factor of $x^2 + 2$.

Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are *monic* (i.e., have 1 as highest-degree coefficient).

What patterns do monic FSP polynomials satisfy?

To make progress on this topic, investigate the following questions and justify your answers.

(a) Are x and $x - 1$ the only monic FSP polynomials of degree 1?

(b) List all the monic FSP polynomials of degree 2.
To start, note that x^2, $x^2 - 1$, $x^2 - x$, and $x^2 + x + 1$ are on that list.
Some of them are products of FSP polynomials of smaller degree. For instance, x^2 and $x^2 - x$ arise from degree 1 cases. However, $x^2 - 1$ and $x^2 + x + 1$ are new, not expressible as a product of two smaller FSP polynomials.
Which terms in your list of degree 2 examples are new?

(c) List all the monic FSP polynomials of degree 3. Which of those are new?
Can you make a similar list in degree 4?

(d) Answers to the previous questions might depend on what coefficients are allowed. List the monic FSP polynomials of degree 3 that have integer coefficients. Separately list those (if any) with complex number coefficients that are not all integers.
Can you make similar lists for degree 4?
Are there examples of monic FSP polynomials with real number coefficients that are not all integers?

a. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. By the Fundamental Theorem of Algebra, either real or non-real, there would be an n number of roots of $f(x)$. Let the roots be $x_1, x_2, \ldots x_n$ .Then, $f(x_1) = f(x_2) = f(x_3) = \cdots = 0$. Since $f(x)$ is a factor of

$f(x^2)$, $f(x^2) = f(x) \times q(x)$. Since it is given that $f(x)$ has the factor-square property,let $x = x_1$, and we get $f({x_1}^2) = f(x_1) = 0$. Since $x = x_1 = x_2 = x_3 = \cdots = x_n$,

$f({x_1}^2) = f({x_2}^2) = f({x_3}^2) = \cdots f({x_n}^2) = 0$. Therefore, we can say that

${x_1}^2, {x_2}^2, {x_3}^2, \ldots, {x_n}^2$ are roots of the function $f(x)$. By the Fundamental Theorem ofAlgebra, $f(x)$ can only have a maximum number of n roots. Since the roots are

$x_1, x_2, x_3, \ldots, x_n$ and ${x_1}^2, {x_2}^2, {x_3}^2, \ldots, {x_n}^2$, we can say that $x_i = {x_j}^2$.

b. A monic FSP polynomial of degree 1 can be expressed as $f(x) = x - a$. Then, $f(x^2) = x^2 - a = (x - a)q(x)$. From this, we know that the zero of $f(x^2)$ is $x = a$. By puttingin this value into the function $f(x^2) = x^2 - a$, we get $a^2 - a = a(a - 1) = 0$.

Therefore, the only two values of a that are satisfactory of this equation is 0 and 1. Knowing the values of a, we can say that x and $x - 1$ are the only monic FSP polynomials of degree 1.

A monic FSP polynomial of degree 2 can be expressed as $f(x) = x^2 + ax + b$.

Then, $f(x^2) = x^4 + ax^2 + b = (x^2 + ax + b)q(x)$.

From this, we know that the zeros of $f(x^2)$ is $x = (-a+\sqrt{a^2-4b})/2$, $(-a-\sqrt{a^2-4b})/2$

However, finding the values of a by putting in these values into the equation $x^4 + ax^2 + b = 0$ would be too complicated. Therefore, this problem needs to be solved with a different approach.

As $x_i = x_j^2$, we can let $rcis\theta_1 = (rcis\theta_2)^2$. Then, $rcis\theta_1 = r^2cis2\theta_2$, and $r = r^2$. Therefore, $r = 0,1$. According to the Fundamental Theorem of Algebra, either real or non-real, any quadratic function would have two roots. Cases can be divided in terms ofmagnitudes of roots.

1. Both roots have magnitudes of 0:
   This would mean that the two roots $f(x)$ would equal to 0. $f(x) = x^2$.
2. Roots have magnitudes of 0, 1:
   This would mean that the two roots $f(x)$ would equal to 0 and $cis\theta$, respectively.

   $cis\theta = cis2\theta$

   $\theta \equiv 2\theta(mod\ 2\pi)$

   $-\theta \equiv 0(mod\ 2\pi)$

   $\theta \equiv 0(mod\ 2\pi)$,

   As $cis0 = 1$, the two solutions of $f(x) = 0,1$. Therefore, $f(x) = x(x-1)$.
3. Both roots have magnitudes of 1:

a) If both roots are real:

$x = 1,1$ or $x = -1,-1$ or $x = -1,1$.

However, $x = -1,-1$ would not make $f(x)$ a FSP polynomial. $(x + 1)^2$ is not afactor of $(x^2 + 1)^2$.

$\therefore x = 1,1$ or $x = -1,1$.

b) If both roots are complex conjugates:

1) $x = cis\theta$

$cis\theta =$

$cis2\theta$ $2\theta \equiv$

$\theta(mod\ 2\pi)$

$\theta \equiv 0(mod\ 2\pi)$

$x = 1$

$f(x) = (x-1)^2$.

2) $x = cis(-\theta)$

$cis(-\theta) = cis2\theta$

$2\theta \equiv -\theta(mod\ 2\pi)$

$3\theta \equiv 0(mod\ 2\pi)$

By using Vieta's formula, we know that $f(x) = x^2 + x + 1$.

c) A monic FSP polynomial of degree 3 can be expressed as $f(x) = x^3 + ax^2 + bx + c$.

Then, $f(x^2) = x^6 + ax^4 + bx^2 + c = (x^3 + ax^2 + bx + c)q(x)$.

As $x_i = x_j^2$, we can let $rcis\theta_1 = (rcis\theta_2)^2$. Then, $rcis\theta_1 = r^2cis2\theta_2$, and $r = r^2$. Therefore, $r = 0,1$. According to the Fundamental Theorem of Algebra, either real ornon-real, any cubic function would have three roots. Cases can be divided in terms ofthe types of roots.

1. When $f(x)$ has three real roots:

a) When $(r_1, r_2, r_3) = (-1,1,1)$, $f(x) = (x + 1)(x-1)^2$. As $f(x^2) = (x^2 + 1)(x^2 - 1)^2 = (x^2 + 1)(x + 1)^2(x - 1)^2 = f(x)(x +$

$1)(x^2+1)$, $f(x)$ is a FSP polynomial.

b) When $(r_1, r_2, r_3) = (-1,1,0)$, $f(x) = x(x^2-1)$. As $f(x^2) = x^2(x^4-1) = x^2(x^2-1)(x^2+1) = f(x)x(x^2+1)$, $f(x)$ is a FSP polynomial.

c) When $(r_1, r_2, r_3) = (0,1,1)$, $f(x) = x(x-1)^2$. As $f(x^2) = x^2(x^2-1)^2 = x^2(x-1)^2(x+1)^2 = f(x) \cdot x(x+1)^2$, $f(x)$ is a FSP polynomial.

d) When $(r_1, r_2, r_3) = (0,1,0)$, $f(x) = x^2(x-1)$. As $f(x^2) = x^4(x^2-1) = x^4(x-1)(x+1) = f(x)x^2(x+1)$, $f(x)$ is a FSP polynomial.

e) When $(r_1, r_2, r_3) = (0,0,0)$, $f(x) = x^3$. As $f(x^2) = x^6 = f(x)x^3$, $f(x)$ is a FSPpolynomial.

f) When $(r_1, r_2, r_3) = (1,1,1)$, $f(x) = (x-1)^3$. As $f(x^2) = (x^2-1)^3 = (x-1)^3(x+1)^3 = f(x)(x+1)^3$, $f(x)$ is a FSP polynomial.

2. When $f(x)$ has one real root and two complex conjugates:

a) When $(r_1, r_2, r_3) = (0, cis(\theta), cis(-\theta))$:

$i)$ $2\theta \equiv \theta(mod\ 2\pi)$:

$\theta \equiv 0(mod\ 2\pi)$

$\theta = 2\pi n$

$\theta = 0$

When $\theta = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta)$ and $cis(-\theta)$ are complex conjugates."

$2\theta \equiv -\theta(mod\ 2\pi)$:

$3\theta \equiv (mod 2\pi)$

$3\theta = 2\pi n$

$\theta = 2/3\pi n$

$\theta = 0, 2\pi/3, 4\pi/3$

- When $\theta = 0$:

  $\theta = 0, 2\,\theta/3, 4\,\theta/3$

  As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta)$ and $cis(-\theta)$ are complex conjugates".

When $\theta = \frac{2\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (0, 1, cis\left(\frac{2\pi}{3}\right), cis\left(-\frac{2\pi}{3}\right))$.

$f(x) = x(x-1)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right) = x(x-1)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = x(x-1)(x^2+x+1) = x^4 - x.$

When $\theta = \frac{4\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (0, 1, cis\left(\frac{4\pi}{3}\right), cis\left(-\frac{4\pi}{3}\right))$.

$f(x) = x(x-1)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right) = x(x-1)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = x(x-1)(x^2+x+1) = x^4 - x.$

When $\theta = 4\pi/3$

$(r_1, r_2, r_3, r_4) = (0, 1, cis\left(\frac{4\pi}{3}\right), cis\left(-\frac{4\pi}{3}\right))$.

$f(x) = x(x-1)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right) = x(x-1)\left(x + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(x + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = x(x-1)(x^2+x+1) = x^4 - x.$

When $(r_1, r_2, r_3) = (1, cis(\theta), cis(-\theta))$:

$2\theta \equiv \theta(mod\ 2\pi)$:

When $\theta = 0$:

$\theta \equiv 0(mod\ 2\pi)$

$\theta = 2\pi n$

$\theta = 0$

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta)$ and $cis(-\theta)$ are complex conjugates."

2. $2\theta \equiv -\theta(mod\ 2\pi)$:

$$3\theta \equiv 0 (mod\ 2\pi)$$
$$3\theta = 2\pi n$$
$$\theta = \frac{2}{3}\pi n$$
$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

- When $\theta = 0$:
- As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta)$ and $cis(-\theta)$ are complex conjugates".

When $\theta = \frac{2\pi}{3}$:

$(r_1, r_2, r_3) = (1, cis(\frac{2\pi}{3}), cis(-\frac{2\pi}{3})$.

$f(x) = (x-1)(x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)) = (x-1)(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i) = (x-1)(x^2+x+1) = x^3-1$.

When $\theta = \frac{4\pi}{3}$:

$(r_1, r_2, r_3) = (1, cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3})$.

$f(x) = (x-1)(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)) = (x-1)(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i) = (x-1)(x^2+x+1) = x^3-1$.

(b) A monic FSP polynomial of degree 4 can be expressed as $f(x) = x^4 + ax^3 + bx^2 + cx + d$. Then, $f(x^2) = x^8 + ax^6 + bx^4 + cx^2 + d = (x^4 + ax^3 + bx^2 + cx + d)q(x)$.

As $x_i = {x_j}^2$, we can let $rcis\theta_1 = (rcis\theta_2)^2$. Then, $rcis\theta_1 = r^2cis2\theta_2$, and $r = r^2$. Therefore, $r = 0,1$. According to the Fundamental Theorem of Algebra, either real ornon-real, any cubic function would have three roots. Cases can be divided in terms ofthe types of roots.

1. When $f(x)$ has four real roots:

a) When $(r_1, r_2, r_3, r_4) = (1,1,1,1)$, $f(x) = (x-1)^4$. As $f(x^2) = (x^2-1)^4 = (x+1)^4(x-1)^4 = f(x) \cdot (x+1)^4$, $f(x)$ is a FSP polynomial.

b) When $(r_1, r_2, r_3, r_4) = (1,1,1,-1)$, $f(x) = (x-1)^3(x+1)$. As $f(x^2) = (x^2-1)^3(x^2+1) = (x+1)^3(x-1)^3(x^2+1) = f(x) \cdot (x+1)^2(x^2+1)$, $f(x)$ is a FSP polynomial.

c) When $(r_1, r_2, r_3, r_4) = (1,1,1,0)$, $f(x) = x(x-1)^3$. As $f(x^2) = x^2(x^2-1)^3 = x^2(x+1)^3(x-1)^3 = f(x) \cdot x(x+1)^3$, $f(x)$ is a FSP polynomial.

When $(r_1, r_2, r_3, r_4) = (1,1,-1,-1)$, $f(x) = (x-1)^2(x+1)^2$. As $f(x^2) = (x^2-1)^2(x^2+1)^2 = (x+1)^2(x-1)^2(x^2+1)^2 = f(x) \cdot (x^2+1)^2$, $f(x)$ is a FSP polynomial.

e) When $(r_1, r_2, r_3, r_4) = (1,1,-1,0)$, $f(x) = x(x+1)(x-1)^2$. As $f(x^2) = x^2(x^2+1)(x^2-1)^2 = x^2(x^2+1)(x+1)^2(x-1)^2 = f(x) \cdot x(x^2+1)(x+1)$, $f(x)$ is a FSP polynomial.

f) When $(r_1, r_2, r_3, r_4) = (1,1,0,0)$, $f(x) = x^2(x-1)^2$. As $f(x^2) = x^4(x^2-1)^2 = x^4(x+1)^2(x-1)^2 = f(x) \cdot x^2(x+1)^2$, $f(x)$ is a FSP polynomial.

g) When $(r_1, r_2, r_3, r_4) = (1,-1,0,0)$, $f(x) = x^2(x+1)(x-1)$. As $f(x^2) = x^4(x^2+1)(x^2-1) = x^4(x^2+1)(x+1)(x-1) = f(x) \cdot x^2(x^2+1)$, $f(x)$ is a FSP polynomial.

h) When $(r_1, r_2, r_3, r_4) = (1,0,0,0)$, $f(x) = x^3(x-1)$. As $f(x^2) = x^6(x^2-1) = x^6(x+1)(x-1) = f(x) \cdot x^3(x+1)$, $f(x)$ is a FSP polynomial.

i) When $(r_1, r_2, r_3, r_4) = (0,0,0,0)$, $f(x) = x^4$. As $f(x^2) = x^8 = f(x) \cdot x^4$, $f(x)$ is a FSP polynomial.

2. When $f(x)$ has two real roots and two complex conjugates:

a) When $(r_1, r_2, r_3, r_4) = (0, 0, cis(\theta), cis(-\theta))$:

$i)\ 2\theta \equiv \theta (mod\ 2\pi)$:

$\theta \equiv 0 (mod\ 2\pi)$

$\theta = 2\pi n$

$\theta = 0$

When $\theta = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta)$ and $cis(-\theta)$ are complex conjugates."

ii) $2\theta \equiv -\theta (mod\ 2\pi)$:

$$3\theta \equiv 0 (mod\ 2\pi)$$
$$3\theta = 2\pi n$$
$$\theta = \frac{2}{3}\pi n$$
$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

When $\theta = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta)$ and $cis(-\theta)$ are complex conjugates".

When $\theta = \frac{2\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (0, 0, cis(\frac{2\pi}{3}), cis(-\frac{2\pi}{3}))$.

$f(x) = x^2 (x - (-\frac{1}{2} + \frac{\sqrt{3}}{2} i))(x - (-\frac{1}{2} - \frac{\sqrt{3}}{2} i)) = x^2 (x + \frac{1}{2} - \frac{\sqrt{3}}{2} i)(x + \frac{1}{2} + \frac{\sqrt{3}}{2} i) = x^2(x^2 + x + 1) = x^4 + x^3 + x^2.$

When $\theta = \frac{4\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (0, 0, cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3})$.

$f(x) = x^2 (x - (-\frac{1}{2} - \frac{\sqrt{3}}{2} i))(x - (-\frac{1}{2} + \frac{\sqrt{3}}{2} i)) = x^2 (x + \frac{1}{2} + \frac{\sqrt{3}}{2} i)(x + \frac{1}{2} - \frac{\sqrt{3}}{2} i) = x^2(x^2 + x + 1) = x^4 + x^3 + x^2.$

When $(r_1, r_2, r_3, r_4) = (0, 1, cis(\theta), cis(-\theta))$:

$i)\ 2\theta \equiv \theta (mod\ 2\pi)$:

$\theta \equiv 0 (mod\ 2\pi)$

$\theta = 2\pi n$

$\theta = 0$

When $\theta = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta)$ and $cis(-\theta)$ are complex conjugates."

$2\theta \equiv -\theta (mod\ 2\pi)$:

$$3\theta \equiv 0 (mod\ 2\pi)$$
$$3\theta = 2\pi n$$
$$\theta = \frac{2}{3}\pi n$$
$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

When $\theta = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta)$ and $cis(-\theta)$ are complex conjugates".

When $\theta = \frac{2\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (0, 1, cis(\frac{2\pi}{3}), cis(-\frac{2\pi}{3}))$.

$f(x) = x(x-1)(x - (-\frac{1}{2} + \frac{\sqrt{3}}{2} i))(x - (-\frac{1}{2} - \frac{\sqrt{3}}{2} i)) = x(x-1)(x + \frac{1}{2} - \frac{\sqrt{3}}{2} i)(x + \frac{1}{2} + \frac{\sqrt{3}}{2} i) = x(x-1)(x^2 + x + 1) = x^4 - x.$

When $\theta = \frac{4\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (0, 1, cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3}))$.

$f(x) = x(x-1)(x - (-\frac{1}{2} - \frac{\sqrt{3}}{2} i))(x - (-\frac{1}{2} + \frac{\sqrt{3}}{2} i)) = x(x-1)(x + \frac{1}{2} + \frac{\sqrt{3}}{2} i)(x + \frac{1}{2} - \frac{\sqrt{3}}{2} i) = x(x-1)(x^2 + x + 1) = x^4 - x.$

When $(r_1, r_2, r_3, r_4) = (1, 1, cis(\theta), cis(-\theta))$:

$2\theta \equiv \theta (mod\ 2\pi)$:

$\theta \equiv 0 (mod\ 2\pi)$

$\theta = 2\pi n$

$\theta = 0$

When $\theta = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is “$cis(\theta)$ and $cis(-\theta)$ are complex conjugates.”

$2\theta \equiv -\theta (mod\ 2\pi)$:

$$3\theta \equiv 0 (mod\ 2\pi)$$
$$3\theta = 2\pi n$$
$$\theta = \frac{2}{3}\pi n$$
$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

When $\theta = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is “$cis(\theta)$ and $cis(-\theta)$ are complex conjugates”.

When $\theta = \frac{2\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (1, 1, cis(\frac{2\pi}{3}), cis(-\frac{2\pi}{3}))$.

$f(x) = (x-1)^2 (x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)) = (x-1)^2 (x+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i) = (x-1)^2(x^2+x+1) = x^4 - x^3 - x + 1.$

When $\theta = \frac{4\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (1, 1, cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3}))$.

$f(x) = (x-1)^2 (x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)) = (x-1)^2 (x+\frac{1}{2}+\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i) = (x-1)^2(x^2+x+1) = x^4 - x^3 - x + 1.$

When $f(x)$ has four complex conjugates:

$(r_1, r_2, r_3, r_4) = (cis(\theta_1), cis(-\theta_1), cis(\theta_2), cis(-\theta_2))$:

$i)\ 2\theta_1 \equiv \theta_1 (mod\ 2\pi)$:

$\theta_1 \equiv 0 (mod\ 2\pi)$

$\theta_1 = 2\pi n$

$\theta_1 = 0$

When $\theta_1 = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is “$cis(\theta_1)$ and $cis(-\theta_1)$ are complex conjugates.”

$2\theta_1 \equiv -\theta_1 (mod\ 2\pi)$:

$$3\theta_1 \equiv 0 (mod\ 2\pi)$$
$$3\theta_1 = 2\pi n$$
$$\theta_1 = \frac{2}{3}\pi n$$
$$\theta_1 = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

When $\theta_1 = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is “$cis(\theta_1)$ and $cis(-\theta_1)$ are complex conjugates.”

When $\theta_1 = \frac{2\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis(\frac{2\pi}{3}), cis(-\frac{2\pi}{3}), cis(\theta_2), cis(-\theta_2))$.

As $2\theta_1 \equiv \theta_2 (mod\ 2\pi)$,

$(r_1, r_2, r_3, r_4) = (cis(\frac{2\pi}{3}), cis(-\frac{2\pi}{3}), cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3}))$.

$f(x) = ((x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)))^2 = ((x+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i))^2 = (x^2+x+1)^2.$

When $\theta_1 = \frac{4\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3}), cis(\frac{8\pi}{3}), cis(-\frac{8\pi}{3}))$.

$f(x) = ((x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)))^2 = ((x+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i))^2 = (x^2+x+1)^2.$

$2\theta_1 \equiv \theta_2 (mod\ 2\pi)$:

$4\theta_1 \equiv \theta_1 (mod\ 2\pi), -\theta_1 (mod\ 2\pi), 2\theta_1 (mod\ 2\pi), -2\theta_1 (mod\ 2\pi)$

$\theta 1 \equiv 0 (mod\ 2\pi)$

$\theta 1 = 2\pi n$

$\theta 1 = 0$

When $\theta_1 = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is “$cis(\theta_1)$ and

$cis(-\theta_1)$ are complex conjugates."

$2\theta_1 \equiv -\theta_1 (mod\ 2\pi)$:

$$3\theta_1 \equiv 0 (mod\ 2\pi)$$
$$3\theta_1 = 2\pi n$$
$$\theta_1 = \frac{2}{3}\pi n$$
$$\theta_1 = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

When $\theta_1 = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta_1)$ and $cis(-\theta_1)$ are complex conjugates."

When $\theta_1 = \frac{2\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis(\frac{2\pi}{3}), cis(-\frac{2\pi}{3}), cis(\theta_2), cis(-\theta_2))$.

As $2\theta_1 \equiv \theta_2 (mod\ 2\pi)$,

$(r_1, r_2, r_3, r_4) = (cis(\frac{2\pi}{3}), cis(-\frac{2\pi}{3}), cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3}))$.

$f(x) = ((x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)))^2 = ((x+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i))^2 = (x^2+x+1)^2$.

When $\theta_1 = \frac{4\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3}), cis(\frac{8\pi}{3}), cis(-\frac{8\pi}{3}))$.

$f(x) = ((x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)))^2 = ((x+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i))^2 = (x^2+x+1)^2$.

$2\theta_1 \equiv \theta_2 (mod\ 2\pi)$:

$4\theta_1 \equiv \theta_1 (mod\ 2\pi), -\theta_1 (mod\ 2\pi), 2\theta_1 (mod\ 2\pi), -2\theta_1 (mod\ 2\pi)$

When $4\theta_1 \equiv \theta_1 (mod\ 2\pi)$:

$$3\theta_1 \equiv 0 (mod\ 2\pi)$$

$3\theta_1 = 2\pi n$

$\theta_1 = \frac{2}{3}\pi n$

$$\theta_1 = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

When $\theta_1 = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta_1)$ and $cis(-\theta_1)$ are complex conjugates."

When $\theta_1 = \frac{2\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis(\frac{2\pi}{3}), cis(-\frac{2\pi}{3}), cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3}))$.

$f(x) = ((x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)))^2 = ((x+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i))^2 = (x^2+x+1)^2$.

When $\theta_1 = \frac{4\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis(\frac{4\pi}{3}), cis(-\frac{4\pi}{3}), cis(\frac{8\pi}{3}), cis(-\frac{8\pi}{3}))$.

$f(x) = ((x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)))^2 = ((x+\frac{1}{2}-\frac{\sqrt{3}}{2}i)(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i))^2 = (x^2+x+1)^2$.

When $4\theta_1 = -\theta_1 (mod\ 2\pi)$:

$$5\theta_1 \equiv 0 (mod\ 2\pi)$$
$$5\theta_1 = 2\pi n$$
$$\theta_1 = \frac{2}{5}\pi n$$
$$\theta_1 = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$$

When $\theta_1 = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta_1)$ and $cis(-\theta_1)$ are complex conjugates."

When $\theta_1 = \frac{2\pi}{5}$:

$(r_1, r_2, r_3, r_4) = (cis(\frac{2\pi}{5}), cis(-\frac{2\pi}{5}), cis(\frac{4\pi}{5}), cis(-\frac{4\pi}{5}))$.

$f(x) = (x^2 - 2x\cos(\frac{2\pi}{5}) + 1)(x^2 - 2x\cos(\frac{4\pi}{5}) + 1) = x^4 + x^3 + x^2 + x + 1$.

When $\theta_1 = \frac{4\pi}{5}$:

$(r_1, r_2, r_3, r_4) = (cis(\frac{4\pi}{5}), cis(-\frac{4\pi}{5}), cis(\frac{8\pi}{5}), cis(-\frac{8\pi}{5}))$.

$f(x) = (x^2 - 2x\cos(\frac{4\pi}{5}) + 1)(x^2 - 2x\cos(\frac{8\pi}{5}) + 1) = x^4 + x^3 + x^2 + x + 1$.

When $4\theta_1 = 2\theta_1 (mod\ 2\pi)$:

$$2\theta_1 \equiv 0 (mod\ 2\pi)$$
$$2\theta_1 = 2\pi n$$
$$\theta_1 = \pi n$$
$$\theta_1 = 0, \pi$$

When $\theta_1 = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta_1)$ and $cis(-\theta_1)$ are complex conjugates".

When $\theta_1 = \pi$:

$(r_1, r_2, r_3, r_4) = (cis(\pi), cis(-\pi), cis(2\pi), cis(-2\pi))$.

$f(x) = (x^2 - 2x\cos(\pi) + 1)(x^2 - 2x\cos(2\pi) + 1) = (x^2 + 2x + 1)(x^2 + 2x + 1) = x^4 + 4x^3 + 6x^2 + 4x + 1$.

When $4\theta_1 = -2\theta_1 (mod\ 2\pi)$:

$$6\theta_1 \equiv 0 (mod\ 2\pi)$$
$$6\theta_1 = 2\pi n$$
$$\theta_1 = \frac{1}{3}\pi n$$
$$\theta_1 = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$$

When $\theta_1 = 0$:

As $cis(0)$ and $cis(-0)$ is real, this option must be excluded as the assumption is "$cis(\theta_1)$ and $cis(-\theta_1)$ are complex conjugates".

When $\theta_1 = \pi/3$:

When $\theta_1 = \frac{\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis\left(\frac{[illegible]}{3}\right), cis\left(-\frac{[illegible]}{3}\right), cis\left(\frac{[illegible]}{3}\right), cis\left(-\frac{[illegible]}{3}\right))$.

[illegible] $2x\cos\left(\frac{\pi}{3}\right) + 1)$ [illegible] $2x\cos\left(\frac{2\pi}{3}\right) + 1) = x^4 + x^2 + 1$.

When $\theta_1 = \frac{2\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis\left(\frac{[illegible]}{3}\right), cis\left(-\frac{[illegible]}{3}\right), cis\left(\frac{[illegible]}{3}\right), cis\left(-\frac{[illegible]}{3}\right))$.

[illegible] $2x\cos\left(\frac{2\pi}{3}\right) + 1)$ [illegible] $2x\cos\left(\frac{[illegible]}{3}\right) + 1) = x^4 +$ [illegible] $+ 3x^2 +$

$2x + 1$.

When $\theta_1 = \pi$:

$(r_1, r_2, r_3, r_4) = (cis(\pi), cis(-\pi), cis(2\pi), cis(-2\pi))$.

$f(x) = (x^2 - 2x\cos(\pi) + 1)(x^2 - 2x\cos(2\pi) + 1) = x^4 - 2x^2 + 1$.

When $\theta_1 = \frac{4\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis\left(\frac{[illegible]}{3}\right), cis\left(-\frac{[illegible]}{3}\right), cis\left(\frac{[illegible]}{3}\right), cis\left(-\frac{[illegible]}{3}\right))$.

[illegible] $2x\cos\left(\frac{4\pi}{3}\right) + 1)$ [illegible] $2x\cos\left(\frac{[illegible]}{3}\right) + 1) = x^4 +$ [illegible] $+ 3x^2 +$

$2x + 1$.

When $\theta_1 = \frac{5\pi}{3}$:

$(r_1, r_2, r_3, r_4) = (cis\left(\frac{5\pi}{3}\right), cis\left(-\frac{5\pi}{3}\right), cis\left(\frac{10\pi}{3}\right), cis\left(-\frac{10\pi}{3}\right))$.

$f(x) = (x^2 - 2x\cos\left(\frac{5\pi}{3}\right) + 1)(x^2 - 2x\cos\left(\frac{10\pi}{3}\right) + 1) = x^4 + x^2 + 1$.