Application of Complex Numbers in Polar Form

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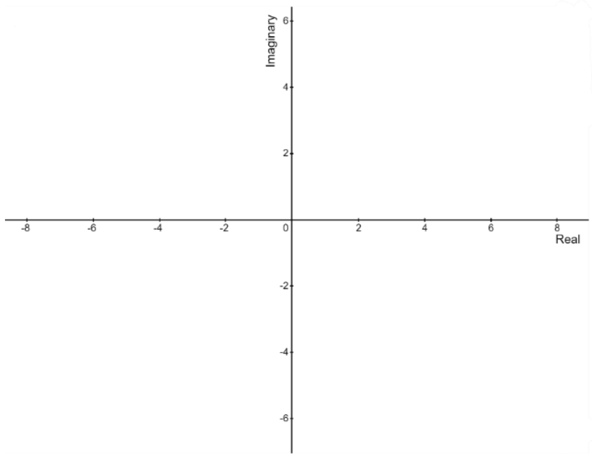
Abstract

A quadratic function has different forms with its own benefits based on different situations—the standard form easily yields the y-intercept, the vertex form easily yields the vertex, and the factored form easily yields the x-intercepts. Likewise, a complex number can be expressed in different forms that are useful for different situations. It is easy to add or subtract two or more complex numbers in standard form, while it is easy to multiply or divide two or more complex numbers in polar form. The first section of this paper explores one of the different forms of complex numbers--the polar form. Then, the next section applies the polar form to find monic factor-square property (FSP) polynomials of different degrees.

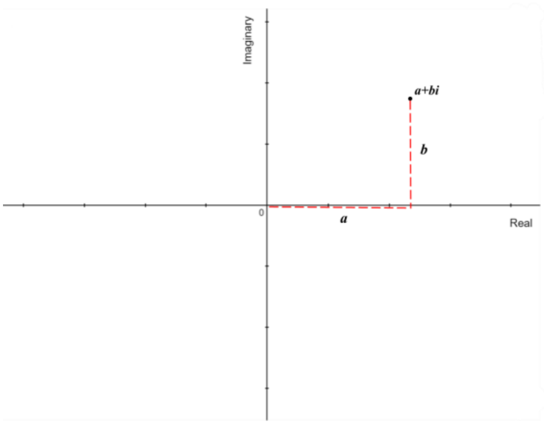
Standard Form of a Complex Number

A complex number is a number that is formed by adding a real number and an imaginary number. Complex numbers are usually expressed in the form of $a + bi$, the standard form, where $i = \sqrt{-1}$. a is called the real part and b is called the imaginary part.

A complex number can be plotted in a coordinate plane with a real axis and an imaginary axis. An example of one is shown below.



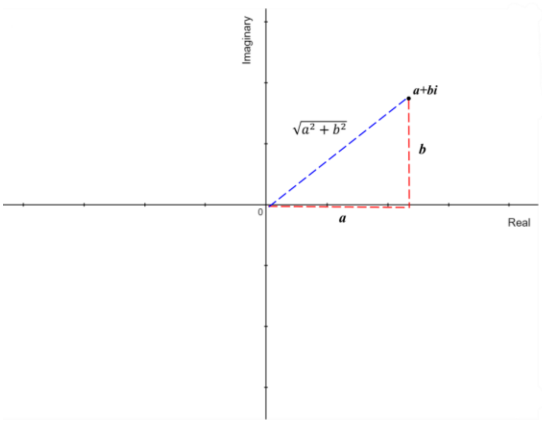
By starting from the origin and moving the point horizontally by a and vertically by b, it is possible to plot the complex number $a + bi$ on the complex plane. In this case, the distance between the point and the imaginary axis would be a and the distance from the real axis would be b.



Properties

Property 1: Absolute value of a complex number

Given a complex number $z = a + bi$, where $i = \sqrt{-1}$, the absolute value of the complex number can be expressed as $|z| = \sqrt{a2 + b2}$. Since the distance between the point and the imaginary axis is a and the distance between the point and the real axis is b, we could create a right triangle by connecting the origin and the point. Therefore, by the Pythagorean theorem, the distance between the origin and z is $\sqrt{a2 + b2}$. Here is a visual representation:



Property 2: Equivalence of Complex Numbers

Given two complex numbers $a + bi$ and $c + di$, where $i = \sqrt{-1}$, $a + bi = c + di$ if and only if $a = c$ and $b = d$. No other cases would work, as either the direction of movement starting from the origin, the distance from the real axis and the imaginary axis, or both would be different.

Property 3: Conjugate Pairs

Given a complex number $z = a + bi$, where $i = \sqrt{-1}$, the conjugate pair of the complex number is expressed as $z = a - b\bar{i}$. There are three properties that follow.

The first property is that $\bar{\bar{z}} = z$. Since the conjugate pair of z can be obtained by changing the sign of the imaginary part, the conjugate pair of $\bar{z}$ would also be obtained by changing the sign of the imaginary part. Doing so would give $a + bi$, which is equal to z.

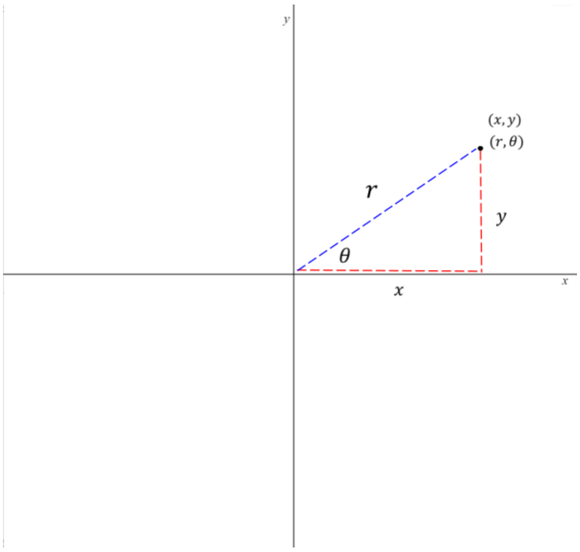
The second property is that given two complex numbers $z = a + bi$ and $w = c + di$, $\overline{z \times w} = \bar{z} \times \bar{w}$. By adding the two complex numbers, $z + w = (a + c) + (b + d)i$. The conjugate pair, $z + w = (\overline{a + c}) - (b + d)i$. Also, $z + w = (a - \bar{b}i) + (\bar{c} - di) = (a + c) - (b + d)i$. Therefore, $\overline{z \times w} = \bar{z} \times \bar{w}$.

The third property is that given two complex numbers $z = a + bi$ and $w = c + di$, $\overline{z \times w} = \bar{z} \times \bar{w}$. Since $z \times w = (a + bi)(c + di) = ac - bd + (ad + cb)i$, $\overline{z \times w} = ac - bd - (ad + cb)i$. Next, $\bar{z} \times \bar{w} = (a - bi)(c - di) = ac - bd - (ad + cb)i$. Therefore, $\overline{z \times w} = \bar{z} \times \bar{w}$.

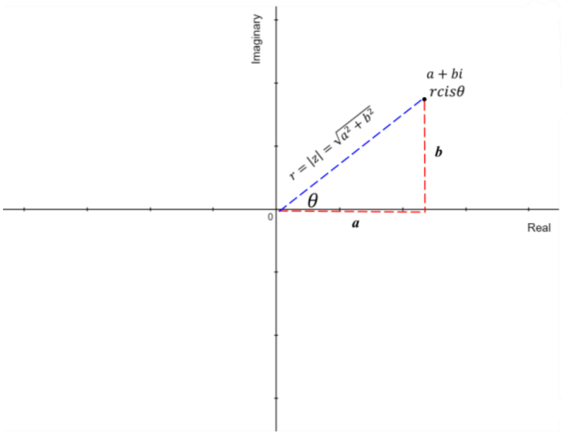
Polar Form of a Complex Number

In order to understand the polar representation of complex numbers, it is necessary to know the fundamentals of polar coordinates. The coordinates of a point on a xy plane can be expressed as $(x,\ y)$, where x is the horizontal position with respect to the origin and y is the vertical position with respect to the origin. The polar coordinates of $(x,\ y)$ is represented as $(r,\ \theta)$, where r is the distance between the point and the origin and θ is the angle from the positive x-axis.

Polar coordinates can be plotted by first plotting the rectangular coordinate (x, y) and then using the following properties: $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, and $\tan\theta = \frac{y}{x}$. Using given information and these properties, we can get the values of r and θ. A visual representation would look like the following image.



The polar coordinates of a complex number can be plotted on a complex plane. Given a complex number $a + bi$, $a = r\cos\theta$, $b = r\sin\theta$, and $\theta = \tan^{-1}\frac{b}{a}$. Using these properties, $a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = rcis\theta$. r is called the modulus (magnitude) and θ is called the argument (angle). Therefore, the polar form of $a + bi$ is $rcis\theta$. As visible in the image below, $r = |z| = \sqrt{a^2 + b^2}$.



To better understand the polar form, consider the following example problems:

Ex1) Find the polar form of the complex number $1 + i$.

In order to find the polar form of a complex number, we must find r and θ. Since $r=\sqrt{a^2+b^2}$, $r = \sqrt{2}$. Also, since $\theta = tan^{-1}\frac{b}{a}$, $\theta = tan^{-1}\ 1 = \frac{\pi}{4}$. Therefore, the polar form of $1 + i$ is $\sqrt{2}cis\frac{\pi}{4}$.

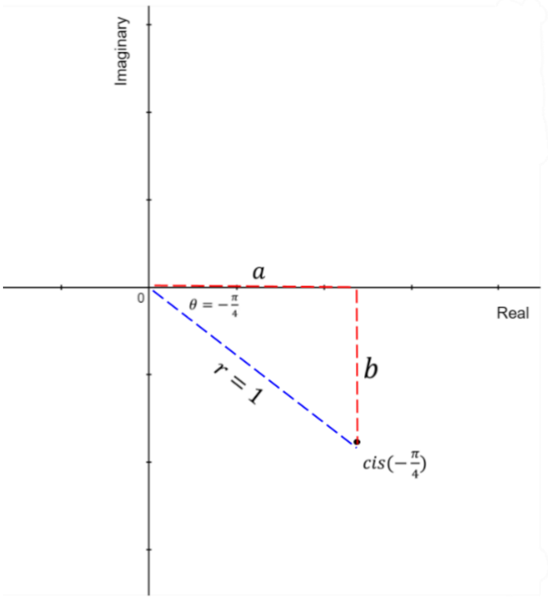
Ex2) Find the polar form of the complex number $-1 + \sqrt{3}i$.

Again, we need to find r and θ. Since $r = \sqrt{a^2 + b^2}$, $r = \sqrt{4} = 2$. Also, since $\theta = tan^{-1}\frac{b}{a} = tan^{-1} - \sqrt{3} = -\frac{\pi}{3}$. Due to the limited range of the inverse tangent function, only a single value of the angle, $-\frac{\pi}{3}$, can be computed. However, it needs to be adjusted to correspond to the angle of the complex number given that is in quadrant II. Therefore, $\theta = \frac{2\pi}{3}$. So, the polar form of the complex number $-1 + 3i$ is $2cis\,\frac{2\pi}{3}$.

We can also convert the polar form of a complex number into the rectangular form. Consider the following example problems:

Ex1) Find the rectangular form of the complex number $cis\left(-\frac{\pi}{4}\right)$.

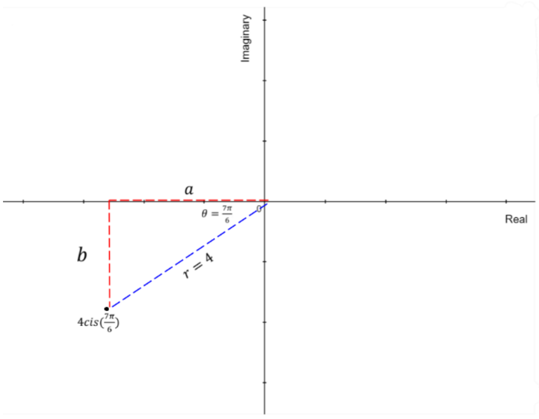
In this case, r = 1 and θ = $-\frac{\pi}{4}$. If we plot this complex number on the complex plane, it would look like the following image:



Since θ = $-\frac{\pi}{4}$, by using special right triangles, $a = \frac{\sqrt{2}}{2}$ and $b = -\frac{\sqrt{2}}{2}$. Therefore, the rectangular form of this complex number is $\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

Ex2) Find the rectangular form of $4cis\left(\frac{7\pi}{6}\right)$.

In this case, r = 4 and θ = $\frac{7\pi}{6}$. Plotting on a complex plane would look like the image below.



The angle formed by the negative real axis and the number is $\frac{\pi}{6}$,. Therefore, by using special right triangles, $a = -2\sqrt{3}$ and $b = -2$. So, the rectangular form of $4cis\left(\frac{7\pi}{6}\right)$ is $-2\sqrt{3} - 2i$.

Addition and Subtraction of Complex Numbers

Given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, the sum/difference of these two complex numbers is $r_1(cos\,\theta_1 + i\,sin\,\theta_1) \pm r_2(cos\,\theta_2 + i\,sin\,\theta_2) = r_1\cos\theta_1 + r_2\cos\theta_2 \pm i(r_1\,sin\,\theta_1 + r_1\,sin\,\theta_1)$. Just like the process of the standard form, the real parts are added or subtracted and the imaginary parts are added or subtracted.

Multiplication and Division of Complex Numbers

Given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(cos\,\theta_2 + i\,sin\,\theta_2)$, the product of these two complex numbers is $r_1r_2(cos\ (\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$. This can be proven by expanding and using trigonometric identities. Here is the full proof:

$$z_1z_2=r_1r_2\cos\theta_1+i\ sin\theta_1)(\cos\ \theta_2+i\sin\ \theta_2)$$

$$=r_1r_2\,(\cos\theta_1\times cos\ \theta_2+cos\theta_1\times i\ sin\theta_2+cos\ \theta_2\times i\ sin\theta_1-sin\theta_1\times sin\theta_2)$$

$$=r_1r_2(\cos\ (\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)).$$

Dividing these complex numbers gives $\frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))$.This can be proven by rationalizing and using trigonometric formulas:

$$\frac{z_1}{z_2}-\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}-\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}$$

$$=\frac{r_1}{r_2}\times\frac{(\cos\theta_1+i\sin\theta_1)(\cos\theta_2-i\sin\theta_2)}{(\cos\theta_2+i\sin\theta_2)(\cos\theta_2-i\sin\theta_2)}$$

$$=\frac{r_1}{r_2}\times\frac{\cos\theta_1\times\cos\theta_2-\cos\theta_1\times i\sin\theta_2+\cos\theta_2-i\sin\theta_1+sin\theta_1\times i\sin\theta_2}{(\cos^2\theta_2+\sin^2\theta_2)}$$

$$=\frac{r_1}{r_2}\times(\cos\theta_1\times\cos\theta_2-\cos\theta_1\times i\sin\theta_2+\cos\theta_2\times i\sin\theta_1+sin\theta_1\times\sin\theta_2$$

$$=\frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\ sin(\theta_1-\theta_2)).$$

Consider the following example problems. Example 1 will be solved with standard form and Example 2 will be solved with polar form.

Ex1) There are two complex numbers $z_1=2+3i\ and\ z_2=3-i.$

a) Find z_1z_2.

$$(2+3i)(3-i)=6-2i+9i+3=9+7i$$

b) Find $\frac{z_1}{z_2}$.

$$\frac{(2+3i)}{3-i}=\frac{(2+3i)(3+i)}{(3-i)+(3+i)}=\frac{6+2\mathrm{i}+9\mathrm{i}+3i^2}{9-i^2}=\frac{3}{10}+\frac{7}{10}i$$

Ex2) There are two complex numbers $z_1-z(cos\frac{\pi}{6}+i\ sin\frac{\pi}{6})$ and $z_2=3\left(cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$.

a) Find z_1z_2.

The multiplication formula states that $z_1z_2=r_1r_2\big(cos(\theta_1{+}\theta_2)+i\ sin(\theta_1{+}\theta_2)\big)$. In this case, $r_1=2,r_2=3,\theta_1=\frac{\pi}{6},\theta_2=\frac{2\pi}{3}$

$$z_1z_2=2\times3\left(co\,s\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{6}+\frac{2\pi}{3}\right)\right)=6\left(cos\frac{5\pi}{6}+i\ sin\frac{5\pi}{6}\right)$$ind $\frac{z_1}{z_2}$.

The division formula states that

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\big(cos(\theta_1{-}\theta_2)+i\ sin(\theta_1{+}\theta_2)\big).$$

By substitution,

$$\frac{z_1}{z_2}=\frac{2}{3}\left(cos\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)+i\ sin\left(\frac{\pi}{6}-\frac{2\pi}{3}\right)\right)$$

$$=\frac{2}{3}\left(cos\left(-\frac{\pi}{2}\right)+i\ sin\left(-\frac{\pi}{2}\right)\right).$$

As seen in the two example problems, using the standard form requires expanding and rationalizing. In order to multiply or divide two, three, four, or more complex numbers, this process would be very complicated. However, using the polar form would facilitate this process because there is no complicated process with the modulus and the argument.

De Moivre's Theorem

Given that the complex number $z = r(\cos\theta + i\sin\theta)$, De Moivre's theorem states that $z^n = r^n(\cos n\theta + i\sin n\theta)$. This means that when a complex number is raised to the power of n, r, the distance between the origin and the complex number, becomes raised to the power of n and θ, the angle formed by the positive real axis and the segment between the origin and the point, becomes multiplied by n.

Here are two example problems:

Ex1) Given that $z = 2\left(cos\,\frac{\pi}{6} + i\,sin\,\frac{\pi}{6}\right)$, find the value of z^6.

De Moivre's theorem states that $z^n = r^n(\cos n\theta + i\sin n\theta)$. In this case, $n = 6$. Therefore, $z^6 = 2^6\left(cos\frac{6\pi}{6} + i\,sin\frac{6\pi}{6}\right) = 64(cos\,\pi + i\,sin\,\pi) = 64 \times -1 = -64$.

Ex2) Given that $z = 2 + 3i$, find the value of z^4

$$z = 2 + 3i = \sqrt{13}\left(\cos\left(tan^{-1}\left(\frac{3}{2}\right)\right) + i\sin\left(tan^{-1}\left(\frac{3}{2}\right)\right)\right)$$

Applying De Moivre's theorem, $z^4 = 169\left(\cos(4tan^{-1}\left(\frac{3}{2}\right) + i\,sin\left(4tan^{-1}\left(\frac{3}{2}\right)\right)\right)$

Before going into the proof of this theorem, it is important to notice this pattern:

$$z^2 = r(cos\theta + i\,sin\theta) \times r(cos\theta + i\,sin\theta)os\theta \times cos\theta + cos\theta \times i\,sin\theta + i\,sin\theta \times cos\theta - sin\theta \times sin\theta)$$

$$= r^2(\cos 2\theta \times i\sin 2\theta) \therefore z_1z_2 = r_1r_2(cos(\theta_1+\theta_2) + i\,sin(\theta_1+\theta_2))$$

$$z^3 = r^2(\cos 2\theta + i\sin 2\theta) \times r(cos\theta + i\,sin\theta)$$

$$= r^3(\cos 2\theta \times cos\theta + \cos 2\theta \times i\,sin\theta + i\sin 2\theta \times cos\theta - \sin 2\theta \times sin\theta)$$

$$= r^3(\cos 3\theta + i\sin 3\theta$$

$$\therefore z_1z_2 = r_1r_2(cos(\theta_1+\theta_2) + i\,sin(\theta_1+\theta_2))$$

$$z^4 = r^3(\cos 3\theta + i\sin 3\theta) \times r(cos\theta + i\,sin\theta)$$

$$= r^4(\cos 3\theta \times cos\theta + \cos 3\theta \times i\,sin\theta + i\sin 3\theta \times cos\theta - \sin 3\theta \times sin\theta)$$

$$= r^4(cos\,4\theta + i\,sin\,4\theta)$$

$\therefore z_1z_2 = r_1r_2\big(cos(\theta_1+\theta_2) + i\,sin(\theta_1+\theta_2)\big)$**ematical Induction and Proof of De Moivre's Theorem**

Mathematical induction is a method to prove a conjecture true. Proof using induction requires two steps. The first step is to prove a conjecture true for base case when $n = k$ (usually $k = 1$). The next step is to assume that the given conjecture, $P(n)$, is true and prove $P(n + 1)$ to be true. If these two steps are successfully proven, the conjecture is proven true for all integer values greater than or equal to k.

Therefore, we can prove De Moivre's Theorem by using mathematical induction.

We have a conjecture that $z^n = r^n(\cos n\theta + i\sin n\theta)$. This conjecture can be proven true for $n = 1$:

$$z^1 = r^1(\cos 1\theta + i\sin 1\theta) \times r(cos\theta + i\,sin\theta).$$

Let $P(n)$, the induction hypothesis, be the statement that $z^n = r^n$ (cos $n\theta$ + i sin $n\theta$). Assuming $P(n)$ is true, we can prove $P(n + 1)$ is true:

$$z^{n+1} = r^n(\cos n\theta + i\sin n\theta) \times r(cos\theta + i\ sin\theta).$$

$$= r^{n+1}(\cos n\theta \times \cos\theta + \cos n\theta \times i\sin\theta + i\sin n\theta \times \cos\theta - \sin n\theta \times \sin\theta)$$

$$= r^{n+1}\big(cos\ (n\theta + \theta) + i\ sin\ (n\theta\ + \theta)\big) = r^{n+1}(cos\ (n+1)\theta + i\ sin\ (n+1)\theta)\ \therefore$$

$$z^{n+1} = r^{n+1}(\cos(n+1)\theta + i\sin(n+1)\theta)$$

Therefore, $z^n = r^n$ (cos $n\theta$ + i sin $n\theta$) is true for all integer values of n.

The nth roots of a complex number

Given that $w^n = z$, where $z = r(\cos\theta + i\sin\theta)$, $n \in \mathbb{N}$, the nth root of a complex number can be represented as $w_k = r^{\frac{1}{n}}\left(cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right)$here k is all integers from 0 to $n-1$. Consider the following examples:

Ex1) Given that $w^6 = 64(\cos\pi + i\sin\pi)$, find the roots.

Since we want to find the 6th root of this complex number, we can represent this complex number as $w^6 = 64(\cos\pi + i\sin\pi)$. Now, using the formula above, we can substitute π for θ, 64 for r, and 6 for n. Also, k = 0, 1, 2, 3, 4, 5.

$$w_0 = 2(\cos\frac{\pi+2\times0\times\pi}{6} + i\ sin\frac{\pi+2\times0\times\pi}{6} = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = 2cis\frac{\pi}{6}$$

$$w_1 = 2(\cos\frac{\pi+2\times1\times\pi}{6} + i\ sin\frac{\pi+2\times1\times\pi}{6} = 2(\cos\frac{3\pi}{6} + i\sin\frac{3\pi}{6} = 2cis\frac{\pi}{2}$$

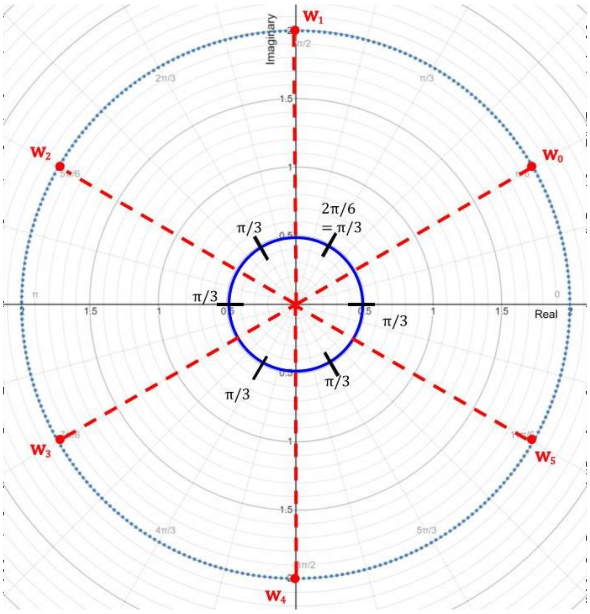
$$w_2 = 2(\cos\frac{\pi+2\times2\times\pi}{6} + i\ sin\frac{\pi+2\times2\times\pi}{6} = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = 2cis\frac{5\pi}{6}$$

$$w_3 = 2(\cos\frac{\pi+2\times3\times\pi}{6} + i\ sin\frac{\pi+2\times3\times\pi}{6} = 2(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = 2cis\frac{7\pi}{6}$$

$$w_4 = 2(\cos\frac{\pi+2\times4\times\pi}{6} + i\ sin\frac{\pi+2\times4\times\pi}{6} = 2(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} = 2cis\frac{3\pi}{2}$$

$$w_5 = 2(\cos\frac{\pi+2\times5\times\pi}{6} + i\ sin\frac{\pi+2\times5\times\pi}{6} = 2(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = 2cis\frac{11\pi}{6}$$

The following image shows the representation of these answers on the complex plane.



As visible from the image above, the angles formed by two consecutive roots are all congruent.

Ex2) Given that $w^8 - i = 0$, find the roots. By the addition property of equality, $z = i$. The polar form of this complex number is $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$, since r = 1 and θ = $\frac{\pi}{2}$. Since we want to find the 8th root of this complex number, we can represent this as z = $w^8=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$. Substituting $\frac{\pi}{2}$ for θ, 1 for r, and 8 for n, we can solve for k = 0, 1, 2, 3, 4, 5, 6, 7.

$$w_0=\cos\left(\frac{\frac{\pi}{2}+\pi}{8}\right)+i\sin\left(\frac{\frac{\pi}{2}+\pi}{8}\right)=\cos\frac{\pi}{16}+i\,sin\frac{\pi}{16}=cis\frac{\pi}{16}$$

$$w_1=\cos\left(\frac{\pi+2\pi}{8}\right)+i\sin\left(\frac{\pi+2\pi}{8}\right)=\cos\frac{5\pi}{16}+i\,sin\frac{5\pi}{16}=cis\frac{5\pi}{16}$$

$$w_2=\cos\left(\frac{\pi+4\pi}{8}\right)+i\sin\left(\frac{\pi+4\pi}{8}\right)=\cos\frac{9\pi}{16}+i\,sin\frac{9\pi}{16}=cis\frac{9\pi}{16}$$

$$w_3=\cos\left(\frac{\frac{\pi}{2}+6\pi}{8}\right)+i\sin\left(\frac{\frac{\pi}{2}+6\pi}{8}\right)=\cos\frac{13\pi}{16}+i\,sin\frac{13\pi}{16}=cis\frac{13\pi}{16}$$

$$w_4=\cos\left(\frac{\frac{\pi}{2}+8\pi}{8}\right)+i\sin\left(\frac{\frac{\pi}{2}+8\pi}{8}\right)=\cos\frac{17\pi}{16}+i\,sin\frac{17\pi}{16}=cis\frac{17\pi}{16}$$

$$w_5=\cos\left(\frac{\frac{\pi}{2}+10\pi}{8}\right)+i\sin\left(\frac{\frac{\pi}{2}+10\pi}{8}\right)=\cos\frac{21\pi}{16}+i\,sin\frac{21\pi}{16}=cis\frac{21\pi}{16}$$

$$w_6=\cos\left(\frac{\frac{\pi}{2}+12\pi}{8}\right)+i\sin\left(\frac{\frac{\pi}{2}+12\pi}{8}\right)=\cos\frac{25\pi}{16}+i\,sin\frac{25\pi}{16}=cis\frac{25\pi}{16}$$

$$w_7=\cos\left(\frac{\frac{\pi}{2}+14\pi}{8}\right)+i\sin\left(\frac{\frac{\pi}{2}+14\pi}{8}\right)=\cos\frac{29\pi}{16}+i\,sin\frac{29\pi}{16}=cis\frac{29\pi}{16}$$

Ex3) Given that $w^3-4\sqrt{3}-4i=0$, find the roots. By the addition property of equality, $z=4\sqrt{3}+4i$. The polar form of this complex number is $8\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$. Since we want the 3rd root of this complex number, we can represent this as $z=w^3=8\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$. Substituting 8 for r, $\frac{\pi}{6}$ for θ, and 3 for n, we can use the formula to calculate the value for k = 0, 1, 2.

$$w_0=2(\cos\frac{\frac{\pi}{6}+0\pi}{3}+i\sin\frac{\frac{\pi}{6}+0\pi}{3})=2(\cos\frac{\pi}{18}+i\,sin\frac{\pi}{18})=2cis\frac{\pi}{18}$$

$$w_1=2(\cos\frac{\frac{\pi}{6}+2\pi}{3}+i\sin\frac{\frac{\pi}{6}+2\pi}{3})=2(\cos\frac{13\pi}{18}+i\,sin\frac{13\pi}{18})=2cis\frac{13\pi}{18}$$

$$w_2=2(\cos\frac{\frac{\pi}{6}+4\pi}{3}+i\sin\frac{\frac{\pi}{6}+4\pi}{3})=2(\cos\frac{25\pi}{18}+i\,sin\frac{25\pi}{18})=2cis\frac{25\pi}{18}$$

The Advantage of Using the Polar Form

The multiplication formula, division formula, and De Moivre's theorem all require the use of polar forms. These theorems facilitate the calculation of complex numbers. As it can be observed by solving the example problems listed above, without having the polar form of a complex number, it takes a very long time and it is very difficult to calculate complex numbers.

In example problem 1 of the multiplication and division section, the numbers were simple and easy to expand. However, if the numbers were more complicated and harder to expand, solving that example would take a very long time. Also, in example problem 2 of the De

Moivre's theorem section, it was not very difficult since the complex number was raised to the 6th power. However, if it was raised to the power of a much higher number, it would be very difficult and time consuming to use the standard form.

On the other hand, if we have the polar form of a complex number, it is much easier to calculate because there are formulas and theorems.

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A polynomial $f(x)$ has the factor-square property (or FSP) if $f(x)$ is a factor of $f(x2)$. For instance, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not. Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are monic (i.e., have 1 as highest degree coefficient).

Setup and Conditions

Let $f(x) = a_n x^n + a_n - 1x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with roots $r_1, r_2, r_3, \ldots, r_n$. If this polynomial has the factor square property, $f(x^2) = q(x)f(x)$, where $q(x)$ is the quotient when $f(x^2)$ is divided by $f(x)$. By substituting r_1 into x, $f(r^{\frac{2}{1}}) = q(r_1)f(r_1)$. Since r_1 is a root of $f(x)$, $f(r_1) = 0$ and $f(r^{\frac{2}{1}}) = 0$. Using the same concept, $r^{\frac{2}{2}}, r^{\frac{2}{3}}, \ldots, r^2$ are all roots of $f(x)$. However, since there can be a maximum of n roots, for each r_i, where $\{i \mid i \in \mathbb{Z}, 1 \le i \le n\}$, there exists a value j, where $\{j \mid j \in \mathbb{Z}, 1 \le j \le n\}$. Each root can be expressed in the polar form of a complex number: $r_k = m_k(\cos\theta_k + i\sin\theta_k)$. Now, the conditions of the magnitudes can be divided into 4 cases.

Case 1: $m_k > 1$

As mentioned before, there needs to exist a $r_j = r^{\frac{2}{k}}$. Using the De Moivre theorem, $r_j = r^{\frac{2}{k}} = m^{\frac{2}{k}}(\cos 2\theta k + i\sin 2\theta k)$. Since $m_k > 1$, $m^{\frac{2}{k}} > m_k$ and $|r_j| > |r_k|$. However, since the modulus keeps on increasing, m_k cannot be greater than 1. Otherwise, there would not exist a $r_j = r^{\frac{2}{k}}$. Therefore, no root has modulus greater than 1.

Case 2: $0 < m_k < 1$

Using the same concept mentioned in case 1, $r_j = r^{\frac{2}{k}} = m^{\frac{2}{k}}(\cos 2\theta k + i\sin 2\theta k$. In this case, since the modulus keeps on decreasing, m_k cannot be a value greater than 0 and less than 1. Otherwise, there would not exist a $r_j = r^{\frac{2}{k}}$. Therefore, no root has $0 < m_k < 1$.

Case 3: $m_k = 0$

If $m_k = 0$, since $0^2 = 0$, there exists a $r_j = r^{\frac{2}{k}}$. Therefore, m_k can be equal to 0. In other words, $r_k = m_k(\cos\theta_k + i\sin\theta_k) = 0(\cos\theta_k + i\sin\theta_k) = 0$.

Case 4: $m_k = 1$

If $m_k = 1$, r_k can be expressed as $\cos\theta_k + i\sin\theta_k$ and $r_j = r^{\frac{2}{k}} = \cos 2\theta_k + i\sin 2\theta_k$. Since $1^2 = 1$, m_k can be equal to 1.

With these 4 cases, all roots have a modulus of 0 or 1. Also, it is important to note that $\theta_j \equiv$

$2\theta_k$ (mod 2π). The reason for using (mod 2π) is, if the difference between the arguments of two complex numbers is a multiple of 2π, the two complex numbers are the same.

Given a polynomial with real coefficients, if $a + bi$ is a root, then $a - bi$ is also a root, which means that if $\cos\theta + i\sin\theta$ is a root, $\cos(-\theta) + i\sin(-\theta)$ is also a root.

a) Are x and $x - 1$ the only monic FSP polynomials of degree 1?

A monic polynomial of degree 1 can be represented as $f(x) = x + a$. In order for $f(x)$ to satisfy FSP, $f(x^2) = f(x) \times q(x)$. Substituting $-a$ for x into this equation yields $f((-a)^2) = f(-a) \times q(-a)$ because $f(-a) = 0$. Thus, $a^2 + a = 0$ and $a = 0$ or -1. Therefore, x and $x - 1$ are the only monic FSP polynomials of degree 1.

b) List all the monic FSP polynomials of degree 2. Some of them are products of FSP polynomials of smaller degree. For instance, x^2 and $x^2 - x$ arise from degree 1 cases. However, $x^2 - 1$ and $x^2 + x + 1$ are "new," not expressible as ax product of two smaller FSP polynomials. Which polynomials are new?

Since the polynomial has degree 2, there can be a maximum of two roots. As mentioned previously, a root can be expressed in the polar form of a complex number:

$r(\cos\theta + i\sin\theta)$. There can be three different cases that satisfy the properties listed above.

Case 1: Both roots have magnitudes of 0.

If both roots are 0, there is only one possible polynomial: x^2.

Case 2: The magnitude of one root equals 0 and that of the other root equals 1.

If $|r| = 1$, the root can be expressed as $\cos\theta + i\sin\theta$. As mentioned previously, there are two conditions to satisfy: $2\theta \equiv \theta$ (mod 2π) and $\theta \equiv -\theta$ (mod 2π). Subtracting θ from both sides of the first condition yields $\theta \equiv 0$ (mod 2π) and adding θ to both sides of the second condition yields $2\theta \equiv 0$(mod 2π). So, $\theta = 2\pi n$ and $2\theta = 2\pi n$. Both of these equations can be satisfied only if $\theta = 0$. Therefore, the only possible polynomial in this case is $x(x - 1) = x2 - x$.

Case 3: Both roots have magnitudes of 1. This case can be divided into two subcases:

3-1: There are two real roots.

There are only two possible roots in this case: 1 and -1. However, if -1 is a root, then 1 also has to be a root. Therefore, there are two possible combinations: $(x - 1)^2 = x^2 - 2x + 1$ and $(x + 1)(x - 1) = x^2 - 1$.

3-2: There are two complex roots which form a conjugate pair.

The two roots can be expressed as $\cos\theta + i\sin\theta$ and $\cos(-\theta) + i\sin(-\theta)$. Taking the first root, in order to satisfy the conditions that $2\theta \equiv \theta$(mod 2π) and $\theta \equiv 0$ (mod 2π), $\theta = 2\pi n$. This would give a real root, which does not count in this case.

Now, taking the second root, the conditions are

$2\theta \equiv -\theta (\text{mod } 2\pi)$ and $3\theta \equiv 0$ (mod 2π), which means that $3\theta = 2\pi n$ and $\theta = \frac{2\pi n}{3}$. So, $\theta = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. By substituting, the first root is $\cos\frac{2\pi}{3} + \text{i}\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\text{I}$, and the second root is $cos\frac{4\pi}{3} + i\,sin\frac{4\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$ So, the polynomial is $\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-1 - \sqrt{3}i\right)\right) = x^2 + x + 1.$

Therefore, the monic FSP polynomials of degree 2 are x^2, $x^2 - x$, $x^2 - 2x + 1$, $x^2 - 1$, and $x^2 + x + 1$. Among these polynomials, $x^2 - 1$ and $x^2 - x + 1$ are new.

c) List all the new monic FSP polynomials of degree 3.

Since the polynomial has degree 3, there can be a maximum of 3 roots. There can be two cases.

Case 1: There are 3 real roots.

We can figure out the polynomials by simply listing out all of the possibilities, using the property that only the numbers 0, 1, -1 would work and that -1 can only be a root if 1 is also a root.

(0, 0, 0) $\rightarrow x^3$. This is not new since it can be represented as $x^2 \times x$.

(0, 0, 1) $\rightarrow x^2(x - 1)$. This is not new since $x2$ and $x - 1$ are from lower degrees.

(0, 1, 1) $\rightarrow x(x - 1)^2$. This is not new since x^2 and $(x - 1)^2$ are from lower degrees.

(0, -1, 1) $\rightarrow x(x^2 - 1)$. This is not new since x and $x^2 - 1$ are from lower degrees.

(-1, 1, 1) $\rightarrow (x + 1)(x - 1)^2$. This is not new since it equals $(x^2 - 1)(x - 1)$.

(1, 1, 1) $\rightarrow (x - 1)^3$. This is not new since it equals $(x - 1)^2(x - 1)$.

(-1, -1, 1) $\rightarrow (x + 1)^2(x - 1)$. This is new since it cannot be represented as a product of two lower degree monic FSP polynomials.

Case 2: There is one real root and two complex roots (one conjugate pair).

As stated before, the only possible real roots are 0 and 1. (-1 can't be a root without 1). The next step is to figure out the complex roots that comply with the properties stated above.

Case 2-1:

$$2\theta \equiv \theta \ (\text{mod } 2\pi)$$

$$\theta \equiv 0 \ (\text{mod } 2\pi)$$

In this case, the root is real. Therefore, there are no complex roots in this case.

Case 2-2:

$$2\theta \equiv -\theta \ (\text{mod } 2\pi)$$

$$3\theta \equiv 0 \ (\text{mod } 2\pi)$$

$$3\theta = 2\pi n, \text{ where } n \in \mathbb{Z}$$

$$\theta = \frac{2\pi n}{3}$$

$$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

If $\theta = \frac{2\pi n}{3}$, the rectangular form of the root is

$-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ and if $\theta = \frac{4\pi}{3}$, the rectangular form of the root is $-1 + \sqrt{3}\, i$. If $\theta = 0$, the root is real and will be one of the roots already listed in the previous degree.

Therefore, the possible combinations of roots are:

$\left(0, -\frac{1}{2}-\frac{\sqrt{3}}{2}i, -\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \rightarrow x(x^2+x+1)$.

This is not new since x and (x^2+x+1) are from lower degrees.

$\left(1, -\frac{1}{2}-\frac{\sqrt{3}}{2}i, -1+\sqrt{3}i\right) \rightarrow (x-1)(x2+x+1)$. This is also not new since $x-1$ and $(x2+x+1)$ are from lower degrees.

The problem also mentions that there are some polynomials with complex coefficients.

If we let α, β, and γ be the three complex roots of the polynomial, by the properties listed above, $\beta = 2\alpha$ and $\gamma = 4\alpha$. So, the roots are α, 2α, and 4α. Now, there can be two different cases: $8\alpha \equiv \alpha \pmod{2\pi}$ and $8\alpha \equiv 2\alpha \pmod{2\pi}$.

If we take the first case, subtracting α from both sides yields $7\alpha \equiv 0 \pmod{2\pi}$. Therefore, $7\alpha = 2\pi n$ and $\alpha = \frac{2\pi n}{7}$. If $n = 1$, the roots are $\frac{2\pi}{7}$, $\frac{4\pi}{7}$, and $\frac{8\pi}{7}$. The list of roots stops there because $\frac{16\pi}{7} \equiv \frac{2\pi}{7} \pmod{2\pi}$. So, the polynomial is $\left(x - cis\left(\frac{2\pi}{7}\right)\right)\left(x - cis\left(\frac{4\pi}{7}\right)\right)\left(x - cis\left(\frac{8\pi}{7}\right)\right)$. By converting to the rectangular form and expanding, the polynomial is $x^3+0.5x^2-0.5x-1+(-1.3x^2-1.3x+0.005056)i$.

If we take the next case, subtracting 2α from both sides yields $6\alpha \equiv 0 \pmod{2\pi}$. So, $\alpha = \frac{2\pi n}{6} = \frac{\pi n}{3}$. If $n = 1$, the roots are $\frac{\pi}{3}, \frac{2\pi}{3}$, and $\frac{4\pi}{3}$. Again, the list stops here because $\frac{8\pi}{3} \equiv \frac{2\pi}{3} \pmod{2\pi}$. Therefore, the polynomial is $\left(x - cis\left(\frac{\pi}{3}\right)\right)\left(x - cis\left(\frac{2\pi}{3}\right)\right)\left(x - cis\left(\frac{2\pi}{3}\right)\right)$. By converting to the rectangular form and expanding,

$\frac{(2x-1)(x^2+x+1)}{2} - \frac{(\sqrt{3}(x^2+x+1)}{2}i = x^3 + \left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)x^2 + \left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x - \left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)$.

This polynomial is new because none of the polynomials from previous degrees have complex coefficients.

Therefore, the new monic FSP polynomials of degree 3 are:

$(x+1)^2(x-1)$

$x^3+0.5x^2-0.5x-1+(-1.3x^2-1.3x+0.005056)i$.

$x^3 + \left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)x^2 + \left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)x - \left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)$.

Can you make a similar list in degree 4?

Case 1: There are 4 real roots.

Using the properties from above, we can list out all of the possible outcomes:

$(0, 0, 0, 0) \rightarrow x4$. This is not new since it can be represented as x^2x^2.

$(0, 0, 0, 1) \rightarrow x^3(x-1)$. This is not new.

$(0, 0, 1, 1) \rightarrow x^2(x-1)^2$. This is not new.

$(0, 1, 1, 1) \rightarrow x(x-1)^3$. This is not new.

$(1, 1, 1, 1) \rightarrow (x-1)^4$. This is not new.

$(0, 0, -1, 1) \rightarrow x^2(x^2-1)$. This is not new.

$(0, -1, 1, 1) \rightarrow x(x-1)^2(x+1)$. This is not new.

$(-1, 1, 1, 1) \rightarrow (x-1)^3(x+1)$. This is not new.

$(-1, -1, 1, 1) \rightarrow (x+1)^2(x-1)^2$. This is not new.

$(-1, -1, -1, 1) \rightarrow (x+1)^3(x-1)$. This is not new.

$(0, -1, -1, 1) \rightarrow x(x+1)^2(x-1)$. This is not new.

Case 2: There are 2 complex roots and 2 real roots.

Case 2-1: The two real roots are 1 and 1.

In this case, the arguments can be 0, 0, θ, or $-\theta$.

Case 2-1-1:

$2\theta \equiv 0 \pmod{2\pi}$ $2\theta = 0 + 2\pi n$

$\theta = \pi n$

This would lead to a real solution.

Case 2-1-2:

$2\theta \equiv \theta \pmod{2\pi}$

$\theta = 2\pi n$

This would also lead to a real solution.

Case 2-1-3:

$2\theta \equiv -\theta \pmod{2\pi}$

$3\theta = 2\pi n$

$\theta = \frac{2\pi n}{3}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

2-1-3-1: $\theta = \frac{2\pi}{3}$

The arguments of the two complex roots are $\frac{2\pi}{3}, -\frac{2\pi}{3} = \frac{4\pi}{3}$

The four roots are 1, 1, $cis\left(\frac{2\pi}{3}\right), cis\left(\frac{4\pi}{3}\right)$.

The polynomial from these roots is:

$(x-1)^2\,(x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right))$

$= (x-1)^2(x^2+x+1)$

2-1-3-2 $\theta = \frac{2\pi}{3}$

The arguments of the roots are $\frac{4\pi}{3}, -\frac{4\pi}{3} = \frac{2\pi}{3}$

This would lead to the same polynomial as case 2-1-3-2.

Case 2-2: The two real roots are 1 and -1. In this case, the arguments are 0, π, θ, $-\theta$.

Case 2-2-1:

$2\theta \equiv 0 \pmod{2\pi}$

Taking case 2-1-1 into consideration, this would lead to a real solution.

Case 2-2-2:

$2\theta \equiv \theta \pmod{2\pi}$

Taking case 2-1-2 into consideration, this would lead to a real solution.

Case 2-2-3:

$2\theta \equiv \pi \pmod{2\pi}$

$2\theta - \pi = 2\pi n$

$2\theta = \pi + 2\pi n$

$\theta = \frac{\pi(2n+1)}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

The polynomial from the resulting roots is:

$(x-1)(x+1)(x-i)(x+i)$

$= (x^2 - 1)(x^2 + 1) = x^4 - 1$. This is a new polynomial.

Case 2-2-4:

$2\theta \equiv -\theta \pmod{2\pi}$

Taking case 2-1-3 into consideration, $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.

The polynomial resulting from these roots is:

$(x-1)(x+1)\left(x - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$

$= (x-1)(x+1)(x^2+x+1)$

Case 2-3: The real roots are 1 and 0.

In this case, the arguments are 0, 0, θ, $-\theta$.

Case 2-3-1:

$2\theta \equiv 0 \pmod{2\pi}$

Taking case 2-1-1 into consideration, this would lead to a real solution.

Case 2-3-2:

$2\theta \equiv \theta \pmod{2\pi}$

Taking case 2-1-2 into consideration, this would lead to a real solution.

Case 2-3-3:

$2\theta \equiv -\theta \pmod{2\pi}$

Taking case 2-1-3 into consideration, $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$.

The polynomial resulting from these roots is:

$x(x-1)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$

$= x(x-1)(x^2 + x + 1)$

Case 2-4: The real roots are 0 and 0.

Case 2-4-1:

$2\theta \equiv 0 \pmod{2\pi}$

Taking case 2-1-1 into consideration, this would lead to a real solution.

Case 2-4-2:

$2\theta \equiv \theta \pmod{2\pi}$

Taking case 2-1-2 into consideration, this would lead to a real solution.

Case 2-4-3:

$2\theta \equiv -\theta \pmod{2\pi}$

Taking case 2-1-3 into consideration, $\theta = \frac{2\pi}{3}$, $\frac{4\pi}{3}$.

The polynomial resulting from these roots is:

$$x^2\left(x-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(x-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)$$

$$= x^2(x^2+x+1$$

Case 3: There are 4 complex roots (2 pairs of complex conjugate roots). In this case, the arguments can be represented as θ, $-\theta$, ϕ, $-\phi$.

Case 3-1

$$2\theta \equiv \theta \ (mod\ 2\pi)$$

This implies that the root with argument θ is real.

Case 3-2

$$2\theta \equiv -\theta \ (mod\ 2\pi)$$

$$3\theta \equiv 0 \ (mod\ 2\pi)$$

$$3\theta = 2\pi n$$

$$\theta = \frac{2\pi n}{3}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

When $\theta = \frac{2\pi}{3}$, the four arguments are $\frac{2\pi}{3}$, $-\frac{2\pi}{3} = \frac{4\pi}{3}$, ϕ, and $-\phi$.

When $\theta = 4\pi$, the four arguments are $\frac{4\pi}{3}$, $-\frac{4\pi}{3} = \frac{2\pi}{3}$, ϕ, and $-\phi$.

Since these two cases lead to the same arguments, the four arguments can be represented as $\frac{2\pi}{3}$, $\frac{4\pi}{3}$,, ϕ, and $-\phi$.

Case 3-2-1

$$2\phi \equiv -\phi \ (mod\ 2\pi)$$

$$\phi = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Case 3-2-1-1

$$\phi = \frac{2\pi}{3}$$

The arguments are $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}$, and $-\frac{2\pi}{3} = \frac{4\pi}{3}$.

The resulting polynomial is $(x2 + x + 1)2 = x4 + 2x3 + 3x2 + 2x + 1$. This polynomial is not new.

Case 3-2-1-2

$$\phi = \frac{4\pi}{3}$$

The arguments are $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3}$, and $-\frac{4\pi}{3} = \frac{2\pi}{3}$.

This would yield the same polynomial as that of case 3-2-1-1.

Case 3-2-2

$$2\phi \equiv \frac{2\pi}{3} (mod\ 2\pi)$$

$$2\phi \equiv \left(\frac{2\pi}{3} + 2\pi n\right)$$

$$\phi = \frac{\pi}{3} + n\pi = \frac{1}{3} + n)\pi$$

$$\phi = \frac{\pi}{3}, \frac{4\pi}{3}$$

Case 3-2-2-1

$$\phi = \frac{\pi}{3}$$

The arguments are $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{\pi}{3} and -\frac{\pi}{3} = \frac{5\pi}{3}$.

The resulting polynomial is $(x^2+x+1)(x^2-x+1)$. This is a new polynomial.

Case 3-2-2-2

$\phi = \frac{4\pi}{3}$

The arguments $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{4\pi}{3}$ $and - \frac{4\pi}{3} = \frac{2\pi}{3}$.

This would lead to the same polynomial as case 3-2-1-2.

Case 3-2-3

$2\phi \equiv \frac{4\pi}{3} (mod\ 2\pi)$

$2\phi - \frac{4\pi}{3} = 2\pi n)$

$\phi = \frac{2\pi n + \frac{4\pi}{3}}{2} = \pi n + \frac{2\pi}{3}$

$\phi = \frac{2\pi}{3}, \frac{5\pi}{3}$

Case 3-2-3-1

$\phi = \frac{2\pi}{3}$

The arguments are $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}$ $and - \frac{2\pi}{3} = \frac{4\pi}{3}$.

This case would yield the same polynomial as case 3-2-1-2.

Case 3-2-3-2

$\phi = \frac{5\pi}{3}$

The arguments are $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$ $and - \frac{5\pi}{3} = \frac{\pi}{3}$.

This case would yield the same polynomial as case 3-2-2-1.

Case 3-3

$2\theta \equiv \phi\ (mod\ 2\pi)$

The four arguments are -2θ, 2θ, $-\theta$, and θ.

Case 3-3-1

$4\theta \equiv -2\theta\ (mod\ 2\pi)$

$6\theta \equiv 0\ (mod\ 2\pi)$

$6\theta = 2\pi n$

$\theta = \frac{\pi n}{3}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

3-3-1-1: $\theta = \frac{\pi}{3}$

The four arguments are $\frac{\pi}{3}$, $-\frac{\pi}{3} = \frac{5\pi}{3}, \frac{2\pi}{3}$ $and - \frac{2\pi}{3} = \frac{4\pi}{3}$

These arguments would yield the same polynomial as that of case 3-2-2-1.

3-3-1-2: $\theta = \frac{2\pi}{3}$

The four arguments are $\frac{2\pi}{3}, -\frac{2\pi}{3} = \frac{4\pi}{3}, \frac{4\pi}{3} - \frac{4\pi}{3} = \frac{2\pi}{3}$.

These arguments would yield the same polynomial as that of case 3-2-1-2.

3-3-1-3: $\theta = \frac{4\pi}{3}$

The four arguments are $\frac{4\pi}{3}, -\frac{4\pi}{3} = \frac{2\pi}{3}, \frac{8\pi}{3} = \frac{2\pi}{3}, \frac{8\pi}{3} = \frac{4\pi}{3}$.

These arguments would yield the same polynomial as that of case 3-2-1-2.

3-3-1-4: $\theta = \frac{5\pi}{3}$

The four arguments are $\frac{5\pi}{3}, -\frac{5\pi}{3} = \frac{\pi}{3}, \frac{10\pi}{3} =$

$\frac{4\pi}{3}-\frac{10\pi}{3}=-\frac{4\pi}{3}=\frac{2\pi}{3}$.

These arguments would yield the same polynomial as that of case 3-2-2-1.

Case 3-3-2

$4\theta \equiv 2\theta \ (mod\ 2\pi)$

$2\theta = 2\pi n$

This case would lead to a real solution.

Case 3-3-3

$4\theta \equiv -\theta \ (mod\ 2\pi)$

$5\theta \equiv 0 \ (mod\ 2\pi)$

$5\theta = 2\pi n$

$\theta = \frac{2\pi n}{5}$

$\theta = \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

3-3-3-1: $\theta = \frac{2\pi}{5}$

$\phi = \frac{2\pi}{5}, -\frac{2\pi}{5}$

$cis\left(\frac{2\pi}{5}\right)=\frac{\sqrt{5}-1}{4}-\frac{\sqrt{2(\sqrt{5}+5)}}{4}i$

$cis\left(-\frac{2\pi}{5}\right)=\frac{\sqrt{5}-1}{4}-\frac{\sqrt{2(\sqrt{5}+5)}}{4}i$

$f(x)=(x-\left(\frac{\sqrt{5}-1}{4}-\frac{\sqrt{2(\sqrt{5}+5)}}{4}i\right))(x-\frac{\sqrt{5}-1}{4}-\frac{\sqrt{2(\sqrt{5}+5)}}{4}i)$

$=x^4+\left(1-\sqrt{5}\right)x^3+\frac{7-\sqrt{5}}{2}+\left(1-\sqrt{5}\right)x+1$

$\approx x^4-1.24x^3+2.38x^2-1.24x+1$

3-3-3-2: $\theta = \frac{4\pi}{5}$

$\phi = \frac{4\pi}{5}, -\frac{4\pi}{5}$

$cis\left(\frac{4\pi}{5}\right)=\frac{-\left(\sqrt{5}+1\right)}{4}-\frac{\sqrt{-2(\sqrt{5}-5)}}{4}i$

$cis\left(-\frac{4\pi}{5}\right)=\frac{-\left(\sqrt{5}+1\right)}{4}-\frac{\sqrt{-2(\sqrt{5}-5)}}{4}i$

$f(x)=(x-\left(\frac{-\sqrt{5}+1}{4}-\frac{\sqrt{-2(\sqrt{5}-5)}}{4}i\right))(x-(\frac{-\sqrt{5}+1}{4}-\frac{\sqrt{-2(\sqrt{5}-5)}}{4}i$

$\approx x^4+3.23x^3+4.62x^2+3.24x+1$

3-3-3-3: $\theta = \frac{6\pi}{5}$

$\emptyset = \frac{6\pi}{5}, -\frac{6\pi}{5}$

$cis\left(\frac{6\pi}{5}\right)=\frac{-\left(\sqrt{5}+1\right)}{4}-\frac{\sqrt{-2(\sqrt{5}-5)}}{4}i$

$cis\left(-\frac{6\pi}{5}\right)=\frac{-\left(\sqrt{5}+1\right)}{4}-\frac{\sqrt{-2(\sqrt{5}-5)}}{4}i$

The polynomial would be the same as that of case 3-3-3-1.

3-3-3-4: $\theta = \frac{8\pi}{5}$

$\phi = \frac{8\pi}{5}, -\frac{8\pi}{5}$

$cis\left(\frac{8\pi}{5}\right)=\frac{\sqrt{5}-1}{4}-\frac{\sqrt{2(\sqrt{5}+5)}}{4}i$

$cis\left(-\frac{8\pi}{5}\right)=\frac{\sqrt{5}-1}{4}-\frac{\sqrt{2(\sqrt{5}+5)}}{4}i$

The polynomial would be the same as that of case 3-3-3-2.

Case 3-3-4

$$4\theta \equiv \theta \ (mod\ 2\pi)$$

$$3\theta \equiv 0 \ (mod\ 2\pi)$$

$$3\theta = 2\pi n$$

$$\theta = \frac{2\pi n}{3}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Both values of θ have already been tested in previous cases.

d) Are there monic FSP polynomials with real number coefficients, but some of those not integers?

Cases 3-3-1, 3-3-2, and 3-3-3 from degree 4 resulted in FSP polynomials with non-integer coefficients. The main difference with these three polynomials is the denominator of the angle before being converted to the rectangular form. These cases were special because they had a denominator of 5. Since it is very difficult to know the exact value of $\sin\left(\frac{n\pi}{5}\right)$ and $\cos\left(\frac{n\pi}{5}\right)$, it is almost impossible to convert to rectangular form by hand. This is why the use of a calculator was necessary for these cases. So, the coefficients of these polynomials are not integers.

Conclusion

This paper was centered on the exploration of properties of complex numbers. The exploration of the polar form of complex numbers and its properties led to the next focus of the paper: solving a problem from 2021 ROSS. Coming up with the proper "setup and conditions" took some time, as the first few trials faced interesting errors and limitations.

Through the implementation of the polar form with roots of polynomials, the properties eventually became coherent. Coming up with the correct properties made the rest of the problem-solving process much smoother and clearer, as the implication of these properties facilitated the setup of cases and subcases.