Properties and applications of complex numbers in polar form

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Abstract

This paper shows the study of complex numbers in polar form and its properties. This includes De Moivre's theorem, and the nth roots of a complex number. The various ways of expressing a complex number are shown: standard form and polar form are the two prominent methods of expressing complex numbers. Each type of expression has its own characteristics. They hold their individual pros and cons when approaching mathematical problems. The benefits of using the polar form rather than the standard form are explored and shown, while looking deeper into the polar form's characteristics. These characteristics play a huge role in how problems are solved, and its application is shown in this paper.

1. Complex Numbers

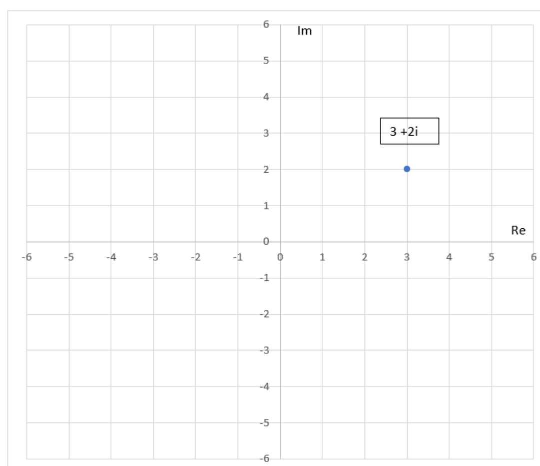
1.1. The Standard Form

A complex number can be expressed in the standard form: $a + bi$

In the standard form, a represents the real part, while b represents the imaginary part of the number. i is a symbol that represents the unit: $\sqrt{-1}$

For instance, the number $3 + 2i$ is a complex number that is expressed in standard form.

This number can be expressed on the **complex plane** (the vertical axis represents the imaginary part while the horizontal axis represents the real part)



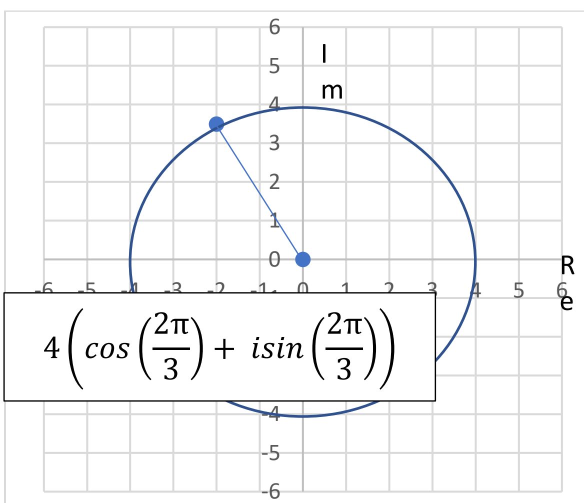
1.2. The Polar Form

A complex number can be expressed in the polar form through the expression $r(\cos(\theta) + i\sin(\theta))$. In the polar form, $\cos(\theta)$ represents the real part while $\sin(\theta)$ represents the imaginary part of the number. r is called the modulus and it represents the magnitude of the complex number. θ is called an argument and it represents the angle between the positive real axis and the complex number.

For instance, the number $4\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$ is a complex number that is expressed in polar form.

In this number, the modulus, or magnitude is equal to 4. Therefore, the point of the complex number must be on the circumference of the circle with radius 4. Also, the argument, or the angle from the positive x axis is 120°. Therefore, the point of the complex number will be in the second quadrant.

This number can also be expressed in a complex plane:



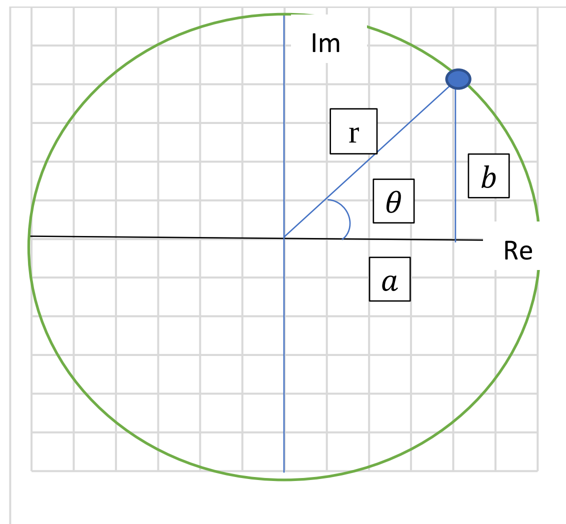
1.3. Relationship between Standard and Polar Form

The polar and standard form are interrelated together in that they can be mathematically converted from one to another.

1.3.1 Standard to Polar Form

Let's consider that we have a complex number $a + bi$ in the standard form.

To covert this to the polar form, we must initially find the magnitude of the number r, then find the angle θ.



Refer to the diagram in the previous page:

$$r^2 = a^2 + b^2$$

In the equation above, because a and b represent the length of the real and imaginary part of a complex number, a and b form a right-angle triangle, and the Pythagorean theorem can be used to find the magnitude of the complex number.

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

In the equation above, we are using tangents to find the angle which can be found using the angle between the number and the positive x real axis. The angle must be measured going counterclockwise from the x axis.

Example) Convert $-2 + 2i$ into the polar form

$$r^2 = a^2 + b^2 = (-2)^2 + (2)^2 = 4 + 4 = 8$$

$$r = 2\sqrt{2}$$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Therefore,

$\theta = \tan^{-1}\left(\frac{2}{-2}\right) = -\frac{\pi}{4}$

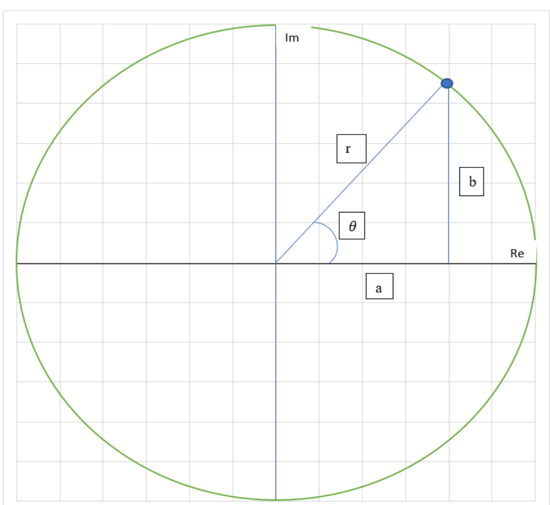
$\tan(\theta) = -\frac{\pi}{4} + \pi n$

The complex number $-2 + 2i$ is in the second quadrant, so we must find the corresponding angle in the second quadrant. Therefore, $\theta = -\frac{\pi}{4} + \pi * 1 = \frac{3\pi}{4}$

Thus,

$z = 2\sqrt{2}(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right))$

1.3.2. Polar to Standard Form



The diagram above presents a complex number on a complex plane. As shown in the graph above, the complex number creates a right-angled triangle on the complex plane.

This represents the following complex number:

$$r(\cos(\theta) + i\sin(\theta))$$

This number can be translated to the standard form.

Because the complex number formed a right triangle,

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{r}$$

$$b = r\sin(\theta)$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{r}$$

$$a = r\cos(\theta)$$

Therefore, $r(\cos(\theta) + i\sin(\theta) = a + bi$

Example) Given that $z = \sqrt{2}(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6})$

Let us covert this into the standard form $a + bi$:

$a = r\cos(\theta) = \sqrt{2}\cos\frac{7\pi}{6} = -\frac{\sqrt{6}}{2}$

$b = r\sin(\theta) = \sqrt{2}\sin\frac{7\pi}{6} = -\frac{\sqrt{2}}{2}$

Thus, $\sqrt{2}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$

2. Product and Quotient of Complex Numbers

2.1. Product

The product of two complex numbers in the **polar form** can be found using the method below.

The product of two complex numbers in the **standard form** can be found using the method below:

$z_1 \times z_2 = (a_1 + b_1i)(a_2 + b_2i)$

$\quad = a_1a_2 + a_1b_2i + a_2b_1i - b_1b_2$

$\quad = a_1a_2 - b_1b_2 + i(a_1b_2 + a_2b_1)$

Two complex numbers z_1 and z_2 are in the polar form:

$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$

$z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$

$z_1 \times z_2 = r_1(\cos(\theta_1) + i\sin(\theta_1)) \times r_2(\cos(\theta_2) + i\sin(\theta_1))$

$= r_1 r_2(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))$

$= r_1 r_2(\cos(\theta_1)\cos(\theta_2) - i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2))$

$= r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$; simplified through trigonometric sum and difference formula

Example:

The following example shows the method of the product of two complex numbers in the polar form:

$z_1 = (-2 + 2i),\ z_2 = (1 - \sqrt{3}i)$

$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$

$r_1 = \sqrt{a^2 + b^2}$

$r_1 = \sqrt{(-2)^2 + (2)^2}$

$r_1 = 2\sqrt{2}$

$\theta_1 = \tan^{-1}\left(\frac{a}{b}\right)$

$\theta_1 = \tan^{-1}\left(\frac{-2}{2}\right)$

$\theta_1 = \frac{3\pi}{4}$

Therefore, $z_1 = 2\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$

$z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$

$r_2 = \sqrt{a^2 + b^2}$

$r_2 = \sqrt{(1)^2 + (-\sqrt{3})^2}$

$r_2 = 2$

$\theta_2 = \tan^{-1}\left(\frac{a}{b}\right)$

$\theta_2 = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$

$\theta_2 = \frac{11\pi}{6}$

Therefore, $z_2 = 2(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6})$

Product of complex numbers formula:

$z_1 \times z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

$z_1 \times z_2 = 2\sqrt{2} * 2(\cos\left(\frac{3\pi}{4} + \frac{11\pi}{6}\right) + i\sin\left(\frac{3\pi}{4} + \frac{11\pi}{6}\right))$

$z_1 \times z_2 = 4\sqrt{2}(\cos\frac{31\pi}{12} + i\sin\frac{31\pi}{12})$

$z_1 \times z_2 = 4\sqrt{2}(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12})$

Using the polar form allows us to find the product of two complex numbers very easily. When multiplying the complex numbers, we can multiply the magnitudes together and the arguments can simply be added together. As we start to multiply more than three complex numbers, it will become easier to use the polar form.

2.2. The Quotient

Quotient of two complex numbers:

$\frac{z_1}{z_2} = \frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i\sin(\theta_2))}$

$= \frac{r_1(\cos(\theta_1) + i\sin(\theta_1))}{r_2(\cos(\theta_2) + i(\sin(\theta_2))} \times \frac{(\cos(\theta_2) - i\sin(\theta_2))}{(\cos(\theta_2) - i\sin(\theta_2))}$

$= \frac{r_1(\cos(\theta_1)\cos(\theta_2) - i\cos(\theta_1)\sin(\theta_2) + i\sin(\theta_1)(\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2))}{r_2(\cos^2(\theta_2) + \sin^2(\theta_2))}$

$= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$; simplified through trigonometric sum and difference formula

Example)

The following example shows the method of the quotient of two complex numbers in the polar form

$z_1 = (1 + i),\ z_2 = (1 + \sqrt{3}i)$

$z_1 = r_1(\cos(\theta_1)\ +\ i(\sin(\theta_1))$

$r_1 = \sqrt{a^2 + b^2}$

$r_1 = \sqrt{(1)^2 + (1)^2}$

$r_1 = \sqrt{2}$

$\theta_1 = \tan^{-1}\left(\frac{a}{b}\right)$

$\theta_1 = \tan^{-1}\left(\frac{1}{1}\right)$

$\theta_1 = \frac{\pi}{4}$

Therefore, $z_1 = \sqrt{2}(\cos\frac{\pi}{4}\ +\ i\sin\frac{\pi}{4})$

$z_2 = r_2(\cos(\theta_2)\ +\ i\sin(\theta_2))$

$r_2 = \sqrt{a^2 + b^2}$

$r_2 = \sqrt{(1)^2 + (\sqrt{3})^2}$

$r_2 = 2$

$\theta_2 = \tan^{-1}\left(\frac{a}{b}\right)$

$\theta_2 = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\theta_2 = \frac{\pi}{6}$

Therefore, $z_2 = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

Quotient of complex numbers formula:

$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}\ (\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right))$

$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$

3. De Moivre's Theorem

From the product of complex numbers, "De Moivre's Theorem" can be derived:

Given that $z = r(\cos(\theta) + i\sin(\theta))$

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

$z_1 \times z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$

Therefore,

$$z^2 = z * z = r^2(\cos(\theta)) + isin(2\theta))$$

$$z^3 = z^2 * z = r^2\big(\cos(2\theta) + isin(2\theta)\big)$$

$$* r\big(\cos(\theta) + isin(\theta)\big)$$

$$= r^3(\cos(3\theta) + isin(3\theta))$$

$z^4 = z^3 * z = r^3\ \big(\cos(3\theta) + isin(3\theta)\big) *$
$r(\cos(\theta) + isin(\theta)$

$$= r^4\big(\cos(4\theta) + isin(4\theta)\big)$$

$z^n = z^{n-1} * z = r^{n-1}(\cos(n-1)\,\theta +$
$isin(n-1)\theta) *\ r(\cos(\theta) + isin(\theta)$

$$= r^n(\cos(n\theta) + isin(n\theta)$$

Proof by induction:

$p(n)$: $(r \times \mathrm{cis}(\theta))^n = r^n(cis(n\theta))$

$When\ n = 1$

$p(1) : r\big(cis(\theta)\big) = r^1(cis(1 \times \theta))$

Therefore, $n = 1$ is true

Assume $n = k$ is true

$p(k)$: $\big(r \times \mathrm{cis}(\theta)\big)^k = r^k(cis(k \times \theta))$

Consider $n = k + 1$

$p(k+1) :\ \big(r \times \mathrm{cis}(\theta)\big)^{k+1}$

$$= r^k\big(cis(k \times \theta)\big) \times (r$$
$$\times\, cis(\theta))$$

$p(k+1) :\ \big(r \times \mathrm{cis}(\theta)\big)^{k+1} = r^k \times r\ \big(cis(k \times \theta + \theta)\big)$; When multiplying two complex numbers, the argument of the product is the sum of the arguments.

$$p(k+1): \left(\mathrm{r} \times \operatorname{cis}(\theta)\right)^{\mathrm{k}+1}$$
$$= r^{k+1}\left(cis(\theta(k+1))\right)$$

Therefore, because $n = 1$ is true and assuming that p(k) is true $n = k + 1$ was proven to be true. Hence by proof by induction, $p(n)$ is true.

The number of complex numbers that are multiplied has a direct relationship with the modulus.

Because the modulus constantly multiplies with each other, the modulus for z^{n} is r^{n}

The argument, on the other hand, forms a sum, proportional to the nth power of the complex number. The argument is constantly added on top of each other, so the argument for z^{n} is $\mathrm{n}\theta$

Example)

In a certain scenario, we are asked to find $(\frac{1}{2}+\frac{1}{2}\mathrm{i})^5$. To do this, we must first convert the complex number into the polar form, then use De Moivre's theorem.

$$z = \frac{1}{2}+\frac{1}{2}\mathrm{i}$$
$$\mathrm{r}^2 = \mathrm{a}^2 + \mathrm{b}^2$$
$$\mathrm{r} = \sqrt{(\tfrac{1}{2})^2 + (\tfrac{1}{2})^2}$$
$$\mathrm{r} = (\tfrac{\sqrt{2}}{2})$$
$$\theta = \tan^{-1}(\tfrac{\mathrm{a}}{\mathrm{b}})$$
$$\theta = \tan^{-1}(\tfrac{\frac{1}{2}}{\frac{1}{2}})$$
$$\theta = \frac{\pi}{4}$$
$$\mathrm{z} = \frac{\sqrt{2}}{2}(\cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right))$$

$$\mathrm{z}^5 = \mathrm{r}^5(\cos(5\theta) + \mathrm{i}\sin(5\theta))$$
$$= \left(\frac{\sqrt{2}}{2}\right)^5 (\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4})$$
$$= \frac{\sqrt{2}}{8}(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i})$$

4. Nth Root of Complex Numbers

Given than z^n = r, given that both z and r are complex numbers and n is a natural number,

The roots of a complex number can be found using the formula below.

$$\mathrm{z_k} = \mathrm{r}^{\frac{1}{\mathrm{n}}}(\cos\left(\frac{\theta+2\mathrm{k}\pi}{\mathrm{n}}\right) + \mathrm{i}\sin\left(\frac{\theta+2\mathrm{k}\pi}{\mathrm{n}}\right)) \quad , \quad \mathrm{k} = 0, 1, 2 \ldots \mathrm{n}-1$$

According to the fundamental theorem of algebra, every polynomial with degree of n has n roots.

Therefore, for instance, if the complex number "z" has power 6, so n=6,

There would be 6 roots in the equation z^n = r

These roots are then evenly spaced out on the complex plane, as shown by the graph below

Ex)

$$\mathrm{z}^6 = \mathrm{i}$$

$$\mathrm{z_k} = \mathrm{r}^{\frac{1}{\mathrm{n}}}\left(\cos\left(\frac{\theta+2\mathrm{k}\pi}{\mathrm{n}}\right) + \mathrm{i}\sin\left(\frac{\theta+2\mathrm{k}\pi}{\mathrm{n}}\right)\right)$$

$$\mathrm{z_0} = \cos\left(\frac{\pi}{12} + \frac{2(0)\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{12} + \frac{2(0)\pi}{6}\right)$$

$$\mathrm{z_1} = \cos\left(\frac{\pi}{12} + \frac{2(1)\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{12} + \frac{2(1)\pi}{6}\right)$$

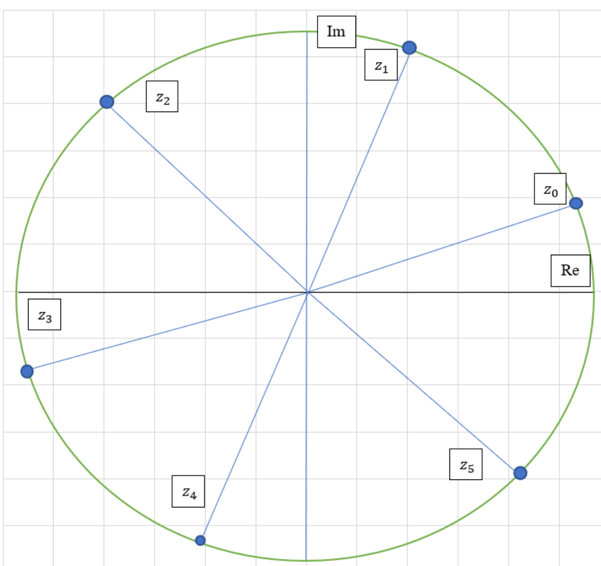
$$\mathrm{z_2} = \cos\left(\frac{\pi}{12} + \frac{2(2)\pi}{6}\right) + \mathrm{i}\sin\left(\frac{\pi}{12} + \frac{2(2)\pi}{6}\right)$$

$$z_3 = \cos\left(\frac{\pi}{12} + \frac{2(3)\pi}{6}\right) + i\sin\left(\frac{\pi}{12} + \frac{2(3)\pi}{6}\right)$$

$$z_4 = \cos\left(\frac{\pi}{12} + \frac{2(4)\pi}{6}\right) + i\sin\left(\frac{\theta}{12} + \frac{2(4)\pi}{6}\right)$$

$$z_5 = \cos\left(\frac{\pi}{12} + \frac{2(5)\pi}{6}\right) + i\sin\left(\frac{\pi}{12} + \frac{2(5)\pi}{6}\right)$$

Each of the six roots of the polynomials are divided into equally distributed 60° spaces.



5. Problem

A polynomial $f(x)$ has the factor-square property (or FSP) if $f(x)$ is a factor of $f(x^2)$. For instance, $g(x) = x - 1$ and $h(x) = x$ have FSP, but $k(x) = x + 2$ does not.

Multiplying by a nonzero constant "preserves" FSP, so we restrict attention to polynomials that are monic (i.e., have 1 as highest-degree coefficient).

What patterns do monic FSP polynomials satisfy? To make progress on this topic, investigate the following questions and justify your answers

Part A.) Are x and x − 1 the only monic polynomials of degree 1 with FSP?

We can represent f(x) with the following:

$f(x) = x + c$

$f(x^2) = (x^2 + c)$

$f(x^2) = (x + c)\big(q(x)\big)$; $x + c$, or $f(x)$, must be a factor of $f(x^2)$ to be an FSP

Therefore, $x^2 + c = (x + c)\big(q(x)\big)$

Using the remainder theorem, we know that when $x = -c$, $(x + c)$ is a factor of $x^2 + c$

When $x = -c$,

$c^2 + c = (-c + c)(q(x))$

$c^2 + c = 0$

$c(c + 1) = 0$

$c = -1, 0$

Therefore,

x and $x - 1$ are the only monic polynomials of degree 1.

Part B.) List all the monic FSP polynomials of degree 2.

To start, note that x^2 , $x^2 - 1$, $x^2 - x$, and $x^2 + x + 1$ are on that list. Some of them are products of FSP polynomials of smaller degree. For instance, x^2 and $x^2 - x$ arise from degree 1 cases. However, $x^2 - 1$ and $x^2 + x + 1$ are new, not expressible as a product of two smaller FSP polynomials.

Which terms in your list of degree 2 examples are new?

$$f(x) = x^2 + ax + b$$
$$f(x^2) = x^4 + ax^2 + b$$
$$x^4 + ax^2 + b = (x^2 + ax + b)(q(x))$$
$$x^2 + ax + b = 0$$
$$x = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$
$$x^4 + ax^2 + b = (x^2 + ax + b)\big(q(x)\big) = 0$$
$$(\frac{-a \pm \sqrt{a^2 - 4b}}{2})^4 + a(\frac{-a \pm \sqrt{a^2 - 4b}}{2})^2 + b = 0$$

When $x = \frac{-a+\sqrt{a^2-4b}}{2}$

Binomial theorem:

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

The coefficients of the binomial theorem are found through combinations or the pascal triangle.

Let us first expand this part of the equation:

$$(\frac{-a + \sqrt{a^2 - 4b}}{2})^4$$

Using the binomial theorem, we can expand the top part of the term

$$\frac{a^4 + 4(-a)^3\sqrt{a^2 - 4b} + 6(-a)^2\sqrt{a^2 - 4b}^2 + 4(-a)\sqrt{a^2 - 4b}^3 + \sqrt{a^2 - 4b}^4}{2^4}$$

$$\frac{a^4 - 4a^3\sqrt{a^2 - 4b} + 6a^4 - 24a^2b - 4a(a^2 - 4b)^{\frac{3}{2}} + a^4 - 8a^2b + 16b^2}{2^4}$$

Let us then expand the next part of the equation:

$$a(\frac{-a + \sqrt{a^2 - 4b}}{2})^2$$

We can expand the top part of the equation

$$\frac{a\left(-a + \sqrt{a^2 - 4b}\right)^2}{2^2}$$

$$\frac{a\left(a^2 + 2(a)\left(\sqrt{a^2 - 4b}\right) + a^2 - 4b\right)}{2^2}$$

We can then combine those two parts to create the following:

$$\frac{8a^4 - 4a^3\sqrt{a^2-4b} - 32a^2b - 4a\left(a^2-4b\right)^{\frac{3}{2}} + 16b^2}{2^4} + \frac{a(a^2 - a\sqrt{a^2-4b} - 2b)}{2} = 0$$

$$\frac{2^2(2a^4 - a^3\sqrt{a^2 - 4b} - 8a^2b - a(a^2 - 4b)^{\frac{3}{2}} + 4b^2)}{2^4} + \frac{a^3 - a^2\sqrt{a^2 - 4b} - 2ab}{2} = 0$$

$$\frac{2a^4 - a^3\sqrt{a^2 - 4b} - 8a^2b - a(a^2 - 4b)^{\frac{3}{2}} + 4b^2}{4} + \frac{a^3 - a^2\sqrt{a^2 - 4b} - 2ab}{2} = 0$$

Solving the problem using remainder theorem and binomial expansion, the problem becomes very complex and difficult to solve.

It is therefore in our best interest to resort to other methods:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Let us consider that $r_1,\ r_2, \ldots . r_n$

An FSP is when $f(x^2) = f(x) * q(x)$

Given that, $x = r_1$

$$f({r_1}^2) = f(r_1)q(r_1) = 0$$

Therefore, $f({r_1}^2) = 0$,

And if the roots of $f(x)$ are $r_1,\ r_2, \ldots . r_n$, then ${r_1}^2,\ {r_2}^2, \ldots . {r_n}^2$ are also roots of $f(x)$

Some roots therefore MUST overlap, and ${r_i}^2 = r_j$

If a root of $f(x)$ is $r_n = m_n(\cos\theta + i\sin\theta)$

Using the De Moivre's Theorem, we can devise that

$${r_n}^2 = {m_n}^2(\cos 2\theta + i\sin 2\theta)$$

Therefore,

$$m_n = {m_n}^2$$
$$m_n - {m_n}^2 = 0$$
$$m_n(m_n - 1) = 0$$
$$m_n = 0{,}1$$

This shows that the magnitude of the root must be either 0 or 1. This also shows that if there is a root with an argument of θ, then there must also be a root with an argument of 2θ

When f(x) is in degree two, we can. Split this into two different cases: when the roots of f(x) have two real numbers, or when f(x) has roots that are complex conjugate pairs:

-1 has an argument of 180°. If it is multiplied by 2, then it becomes 360° or 0°. Therefore 1, which has an argument of 0°, must also exist as the other root when -1 is a root.

When there are two real roots, there are the following possibilities:

The roots are (0, 0) meaning $f(x) = x^2$

The roots are (1, 1) meaning $f(x) = (x-1)^2 = x^2 - 2x + 1$

The roots are (1, -1) meaning $f(x) = (x+1)(x-1) = x^2 - 1$

The roots are (0, 1) meaning. $f(x) = x(x-1) = x^2 - x$

When there are two complex roots, a conjugate pair would be found:

Complex conjugates have a relationship such that when the argument of one complex number is θ, the other is $-\theta$. This is because when one of the complex numbers are $a + bi$, the other is $a - bi$. This means that only the sign imaginary part of the complex number is changed. Therefore, in polar form, the two complex numbers must have a relationship in which they are reflected across the real axis.

Therefore:

$z_1 = r_1(\cos\theta + i\sin\theta)$

$z_2 = r_2(\cos(-\theta) + \sin(-\theta))$

Because they are FSP's, we know that

$-\theta = 2\theta \pm 360n$

$3\theta = \pm 360n$

$\theta = \pm 120n$

$\theta = 0°, 120°, 240°$

In this case, 0° is a real number, and does not fit under the category of "two complex roots". Therefore, it can be ignored for this part of the question.

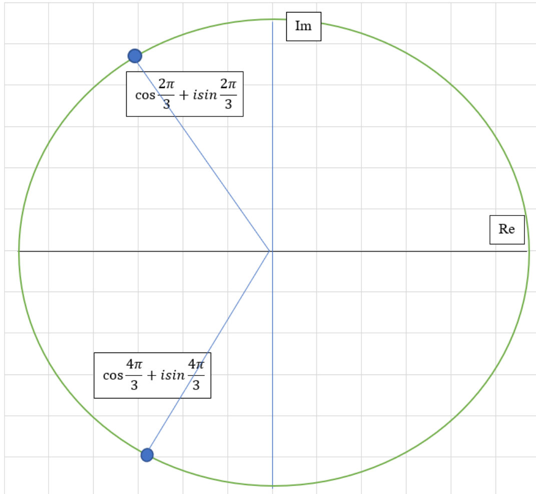
In the case of 120°, $-\theta = -120°$, and 120° fits under the condition $\theta = -\theta$. Because $-120° = 240°$, and 120° and 240° are complex conjugates, $\theta = 120°, 240°$ could be considered a solution for this question.

In the case of 240°, $-\theta = -240°$, and 240° fits under the condition $\theta = -\theta$. Because $-240° = 120°$, and 120° and 240° are complex conjugates, $\theta = 120°, 240°$ could be considered a solution for this question.

$z_1 = \cos 120° + i \sin 120°$

$z_2 = \cos 240° + i \sin 240°$

This is shown in the diagram below:



$$f(x) = \left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right) = x^2 + x + 1$$

Part C.1.) List all the new monic FSP polynomials of degree 3.

When f(x) is in degree three, we can split this into two different cases: when the roots of $f(x)$ has three real roots and when 1 real root and two complex roots.

Case 1.) When there are three real roots, there are the following possibilities:

The roots are (0,0,0) meaning $f(x) = x^3$

The roots are (0,1,-1) meaning $f(x) = x(x - 1)(x + 1) = x^3 -$

The roots are (1, 1, 1) meaning $f(x) = (x - 1)^3 = x^3 - 3x^2 + 3x - 1$

The roots are (1, -1, 1) meaning $f(x) = (x + 1)(x - 1)^2 = x^3 - x^2 - x + 1$

The roots are (1, -1, -1) meaning $f(x) = (x + 1)^2(x - 1) = x^3 + x^2 - x - 1$

The roots are (0, 1, 1) meaning $f(x) = x(x - 1)^2 = x^3 - 2x^2 + x$

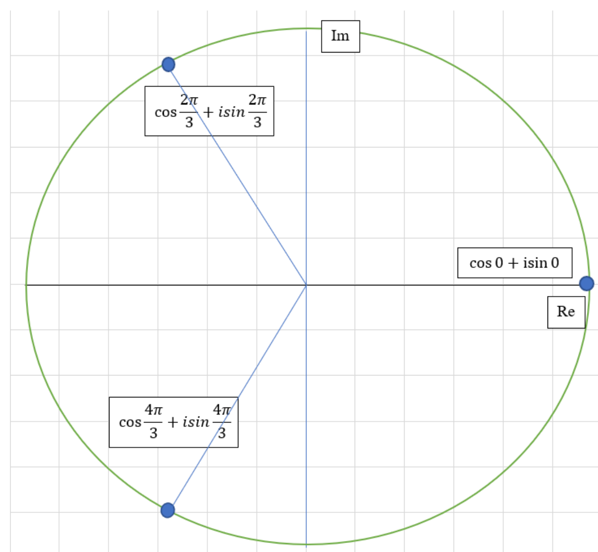
The roots are (0, 0, 1) meaning $f(x) = x^2(x + 1) = x^3 + x^2$

Case 2.) When there are two complex conjugate roots and one real root, we can have the following scenario:

From the reasoning from part B, we know that when there are two complex conjugate roots, the arguments must be 120° and 240°.

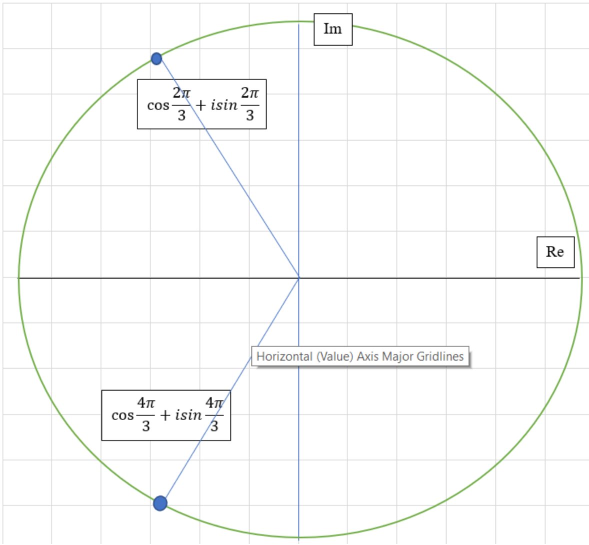
The two possible solutions are when x = 0, 1 and there are two other complex conjugate roots.

When the real root is 1:



$$f(x) = (x - 1)\left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right) = x^3 - 1$$

When the real root is 0:



$$f(x)=x\left(x-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)\left(x-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right)=x^3+x^2+x$$

There is also another possibility of one complex pair and one real root. However, this means that the real root is -1, which is impossible as when a root is -1, another root must be 1.

Part C.2.) List all the new monic FSP polynomials of degree 4.

There are three cases we can create with polynomials of degree 4. When there are four real roots, a pair complex conjugate roots and two real roots, and two pairs of complex conjugate roots.

Case 1.) When there are four real roots, we can create the following possibilities:

The roots are (0,0,0,0) meaning $f(x)=x^4$

The roots are (0,0,0,1) meaning $f(x)=x^3(x-1)=x^4-x^3$

The roots are (0,0,1,1) meaning $f(x)=x^2(x-1)^2=x^4-2x^3+x^2$

The roots are (0,1,1,1) meaning $f(x)=x(x-1)^3=x^4-3x^3+3x^2-x$

The roots are (1,1,1,1) meaning $f(x)=(x-1)^4=x^4-4x^3+6x^2-4x+1$

The roots are (0,0,1, -1) meaning $f(x)=x^2(x-1)(x+1)=x^4-x^2$

The roots are (0, 1, 1, -1) meaning $f(x)=x(x+1)(x-1)^2=x^4-x^3-x^2+x$

The roots are (1,1,1, -1) meaning $f(x)=(x+1)(x-1)^3=x^4-2x^3+2x-1$

The roots are (1,1, -1, -1) meaning $f(x)=(x-1)^2(x+1)^2=x^4-2x^2+1$

The roots are (0,1, -1, -1) meaning $f(x)=x(x-1)(x+1)^2=x^4+x^3-x^2-x$

The roots are (1, -1, -1, -1) meaning $f(x)=(x-1)(x+1)^3=x^4+2x^3-2x-1$

Case 2.) When there are two real roots and a pair of complex roots, we can have the following scenario:

The real roots that fit with the combination of 0° and 0° are the following:

(0, 1), (1, 1), (0, 0)

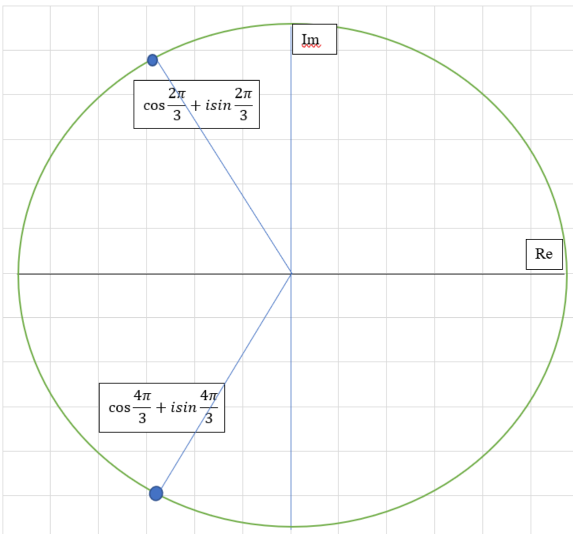
The two complex number conjugate that can be found and fit into this combination is 120°, 240°. The process finding was shown in the scenarios above.

$(0,\ 0,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i)$

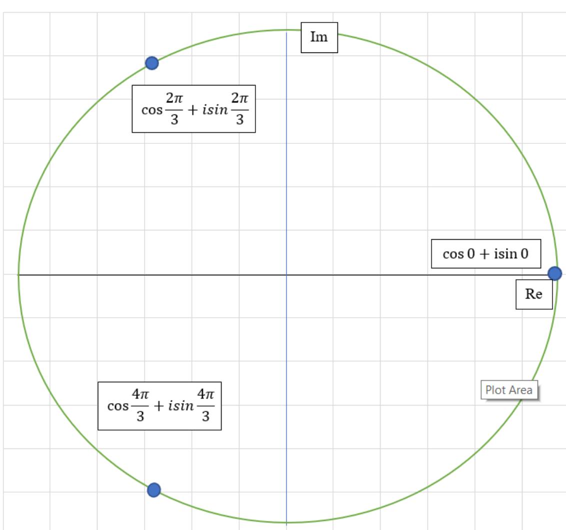
$(0,\ 1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i)$

$\left(1,\ 1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$

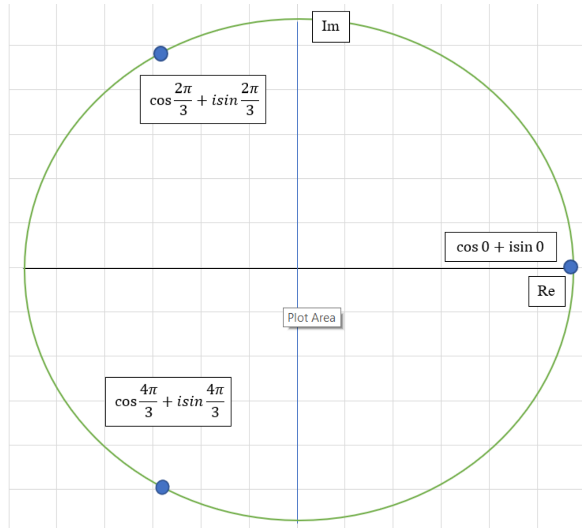
These are shown graphically below:



$$f(x)=x^2\left(x-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)\right)\left(x-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)\right)=x^4+x^3+x^2$$



$$f(x)=(x-1)^2\left(x-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)\right)\left(x-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)\right)=x^4-x^3-x+1$$



$$f(x)=x(x-1)\left(x-\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)\right)\left(x-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)\right)=x^4-x$$

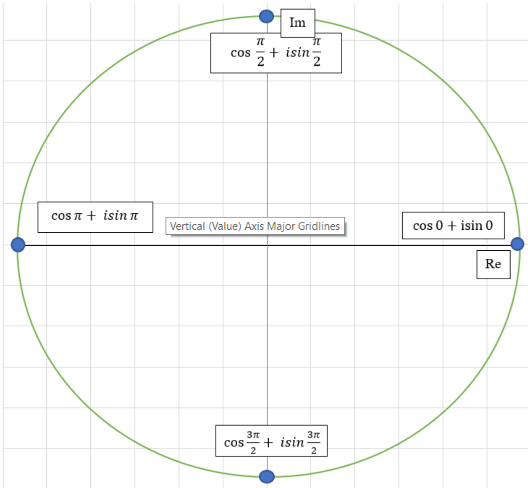
When two of the real roots have an argument of 180° and 0°, the following cases can be found:

(1, -1, i, -i)

$\left(1,\ \text{-}1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$

The cases are represented in the following diagrams below:

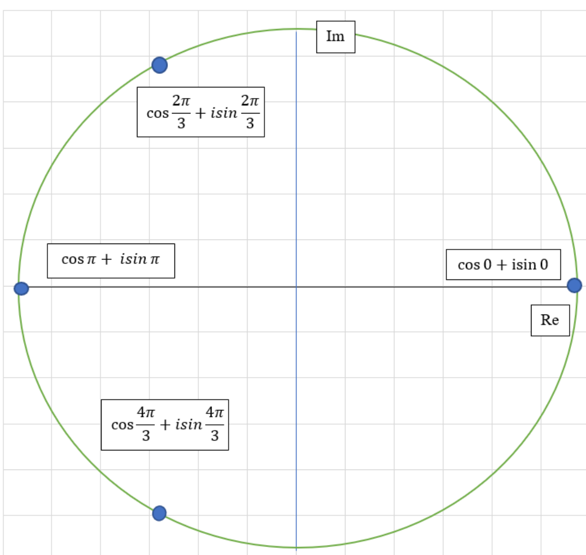
When the real roots are 1, -1:



$$\mathrm{f(x)} = (\mathrm{x}+1)(\mathrm{x}-1)(\mathrm{x}-\mathrm{i})(\mathrm{x}+\mathrm{i})$$

$$\mathrm{f(x)} = (\mathrm{x}^2-1)(\mathrm{x}^2+1)$$

$$\mathrm{f(x)} = \mathrm{x}^4-1$$



$$\mathrm{f(x)} = (\mathrm{x}+1)(\mathrm{x}-1)\left(\mathrm{x}-\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\mathrm{i}\right)\right)\left(\mathrm{x}-\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\mathrm{i}\right)\right) = \mathrm{x}^4+\mathrm{x}^3-\mathrm{x}-1$$

Case 3.) There are also solutions when there are two pairs of complex roots:

Given that the four complex roots have an argument of $\theta, -\theta,\ \phi, -\phi$, as they must be conjugates, we know that one pair, $\theta\ and -\theta$, must have roots of 120° and 240° from the sections above.

We must then find the potential roots of $\phi\ and\ -\phi$. There are four potential cases of finding the pair of conjugate roots.

a. $2\phi = \phi + 2\pi\mathrm{n}$

This means that the roots of ϕ are real numbers and these cases have already been covered from above.

b. $2\phi = -\phi + 2\pi\mathrm{n}$

$$\phi = \frac{2\pi}{3}, \frac{4\pi}{3}$$

The values of $\theta, -\theta,\ \phi, -\phi$ become $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, respectively, and the roots can be expressed as the following:

$$f(x) = \left(\mathrm{x}-\left(-\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}\mathrm{i}\right)\right)^2\left(\mathrm{x}-\left(-\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}\mathrm{i}\right)\right)^2$$

$$f(x) = x^4 + 2x^3 + 3x^2 + 2x + 1$$

c. $2\phi = \frac{2\pi}{3} + 2\pi n$

$$\phi = \frac{\pi}{3} + \pi n$$

$$\phi = \frac{\pi}{3}, \frac{4\pi}{3}$$

The values of $\theta, -\theta,\ \phi, -\phi$ become $\frac{2\pi}{3}, \frac{4\pi}{3}, \frac{\pi}{3}, -\frac{\pi}{3}$, respectively, and the roots can be expressed as the following:

$$f(x) = \left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$$

$$f(x) = x^4 + x^2 + 1$$

d. $2\phi = \frac{4\pi}{3} + 2\pi n$

$$\phi = \frac{2\pi}{3} + \pi n$$

This makes the values of $\phi = \frac{2\pi}{3}, \frac{5\pi}{3}$

This is repetitive to the case found in 'c' as $\frac{5\pi}{3} = -\frac{\pi}{3}$, and can be disregarded

Part D.) Are there monic FSP polynomials (of some degree) that have real number coefficients, but some of those coefficients are not integers? Explain your reasoning.

In part A, B, and C, the problem was solved considering that the coefficients were real numbers. There was no restriction to the method that caused the coefficient values to specifically become integers. All cases regarding rational numbers were considered, and the coefficient values were all integers. Therefore, there are no FSP polynomials that have real number coefficients that are not integers.

6. Conclusion

To solve problem, I gave multiple attempts to use the standard form. However, upon several trials, I was able to understand that there were limitations within using the standard form, and the use the polar form was necessary. While it was possible to solve part A using the standard form, I found that the binomial theorem too stiff in that it did not connect the real and imaginary part of complex numbers. The polar was much more convenient in that it could express the complex number in a more flexible way using arguments. Arguments gave the complex numbers more flexibly in terms of multiplying, dividing, and manipulating, making it much easier to solve the problem.